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To my parents and Joachim

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Zhen Shi

Melbourne

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Chapter 1 Introduction

In this decade, the pension fund industry has experienced two "perfect storms" with both interest rates and stock prices falling dramatically at the same time. Falling interest rates increase values of pension liabilities. Falling equity prices decrease investment return of pension funds. Both sides of their balance sheets are hit hard by bad news from the financial market. Figure 1 shows the S&P 500 index and the U.S. 10-year government bond yield from 1998 to May 2009. The first storm happens at around 2003 and 2004. At the end of 2000, the yield of the U.S. 10-year government bond is above 6%. The yield reduced to about 3-4% in 2003 and 2004. During the same time period, the S&P 500 index drops to 1200 in 2003 from above 2000 at the beginning of this decade. As a result of the “perfect storm”, the total amount of pension plan underfunding in the U.S. is about $354 billion in 2004 increasing from $20 billion in 2000, as indicated in table 1. In 2009, the pension industry is hit by another "perfect storm", where the S&P 500 index drops to about 1300 and the 10-year government bond yield is as low as 3%. The ageing population and the financial market turmoil create enormous challenges to pension funds all over the world.

This thesis focuses on the three major participants in pension finance, namely, pension funds, individuals, and sponsoring companies. In the light of the fragile financial market performance, prudential regulatory rules, including Value-at-Risk (VaR) constraints, are imposed widely all over the world. The purpose of these regulations is to reduce the risk of pension funds and protect pension fund participants. There are no doubts that these regulations will re-shape pension funds' investment strategies. Chapter 2 investigates the optimal investment strategies of a pension fund under the VaR constraint. Defined Benefit and Defined Contribution are the two most common types of pension plans. Individuals with Defined Contributions (DC) pension plan[1] get a lump sum when they retire. They can then decide whether and when to annuitize the lump sum. The annuity income depends on the size of the pension wealth and the interest rate at the annuitization time. Chapter 3 analyzes retirement timing decisions of DC pension plan participants, taking into account the optimal annuitization timing decision. Companies sponsoring underfunded plans are typically required by law to make additional financial contributions to close the funding gap. In the midst of a financial crisis, mandatory contributions will severely tighten financial constraints of sponsoring companies. Chapter 4 develops an optimal investment strategy for a company sponsoring an under-funded pension plan

[1]In Australia, individuals with Defined Benefit (DB) pension plans will also get a lump sum when they retire.

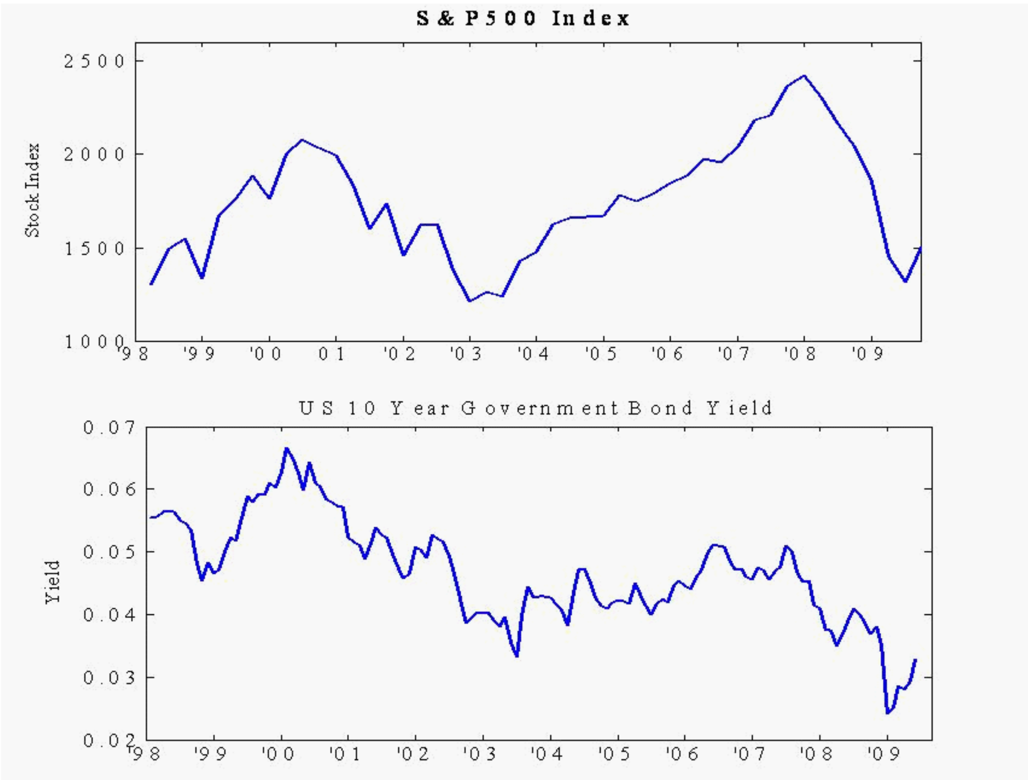


Figure 1: The upper panel of this figure shows the S&P 500 index during 1998 and May 2009. The lower panel of this figure shows the U.S. 10-year government bond yield of the same period. Source: Datastream.

	2000	2001	2002	2003	2004
Number of Underfunded pension Plans	221	747	1058	1051	1108
Pension Underfunding (Dollars in billions)	\$19.91	\$110.94	\$305.88	\$278.99	\$353.73

Table 1: This tables shows the summary of pension underfunding filing. Source: PBGC (2005)

with mandatory contribution requirement, aiming to reduce the impact of mandatory contributions on financial constraints. The following paragraphs provide a more detailed overview.

Chapter 2, *Pension Fund Portfolio Wealth under Short-Term Regulation,* studies economic consequences of a misalignment in the planning horizon between an institutional investor, for example, a pension fund, pursuing long-term investment strategies and a regulator enforcing a Value-at-Risk (VaR) type solvency constraint on the institutional investor on a short-term basis. The smaller the regulatory horizon the more often the investor needs to fulfill the VaR constraint. However, the VaR-constrained investor is only concerned about the probability but not the magnitude of the loss. Therefore, the investor is willing to incur losses in compliance with the VaR constraint. For example, in the case when the VaR horizon is as long as the investment horizon, a VaR-constrained investor keeps the portfolio value above or at the threshold value, e.g. the value of the liability, when the investment environment is favorable but leaves his portfolio completely uninsured in the worst investment environment. By definition, the worst investment environment occurs with probability equals to α which is set by the regulator ex ante. Chapter 2 shows that short-horizon VaR regulation can limit this moral hazard behavior due to the minimum amount of portfolio wealth required to fulfill future VaR constraints. A VaR constraint allows a small probability that a pension fund's portfolio wealth falls below its liability value, while a portfolio insurance constraint requires that probability to be zero. Chapter 2 finds that a short-term VaR constraint can have a similar impact on a pension fund's portfolio wealth as a portfolio insurance constraint. For a 100% funded pension plan, the loss caused by the annual VaR constraint can be as large as a 11% reduction in its current asset value.

Chapter 2 also shows that the investment strategy of a VaR-constrained investor depends on the frequency the VaR constraint is imposed. In the case when the VaR horizon is as long as the investment horizon, the investor under a VaR constraint will first invest in equities as much as an unconstrained investor would do. As the investment environment deteriorates, he will reduce his allocation to equities. However, as the investment environment deteriorates even further, he starts to increase his allocation to equities because there is still a large chance he will end up in a state where the VaR constraint is binding. When the environment becomes really bad, he behaves as if he is not constrained. Compared with the investment strategy of an investor with a long-term VaR constraint, an investor with short-term VaR constraints will (1) decrease allocations to the stock index much faster, (2) not gamble as much, and (3) invest 100% in bonds in bad states. The last two results are due to the fact that the shorter the VaR horizon the more binding it is and the investor has to make sure that the portfolio wealth is kept above a certain wealth level to fulfill all the future VaR constraints. The minimum wealth level also reduces room for gambling.

Chapter 3, *Annuitization and Retirement Timing Decisions*, analyzes retirement timing decisions of Defined Contribution (DC) pension plan participants, taking into account the optimal annuitization timing decision. Individuals' DC wealth shrinks a lot during the current financial crisis. However, recently a survey by the Employee Benefit Research Institute (EBRI) shows that there are not many persons would like to postpone their retirement decision to accumulate more DC wealth. It might be worrying that the individuals will not have sufficient financial wealth to support their retirement lives. DC pension plans generally provide a lump-sum payment at the retirement date. Individuals typically have large freedom to decide when to annuitize their DC wealth after retirement. This freedom allows individuals to benefit from better financial market performance after retirement. Therefore, such a concern might be unnecessary.

Chapter 3 first sets up a retirement decision model and develops a forward looking retirement likelihood measure from this model. The retirement likelihood measure describes the probability that an individual will retire within the next few years. In the model, the individual obtains utility from leisure, labor income before retirement and pension benefit after retirement. The DC pension benefit is the income from the annuity which is bought at the optimal annuitization timing. And then, the retirement likelihood measure is tested with the English Longitudinal Study of Ageing (ELSA) data. In order to assess the predictive power of this model, retirement likelihoods derived from the theoretical setup are compared with retirement decisions observed at the second wave of ELSA for a sample of individuals who were full-time employed at the time of the first wave interviews. It turns out that the theory-motivated retirement likelihood measure is a statistically significant predictor of actual retirement decisions. Moreover, Chapter 3 shows that the proposed retirement likelihood measure is highly correlated with observed retirement ratios across groups of individuals defined by age or wealth.

Chapter 4, *Corporate Investment Strategy and Pension Underfunding Risk*, discusses the optimal investment strategy for a firm sponsoring a Defined Benefit (DB) pension plan. The proposed investment strategy will mitigate the impact of a liquidity shock resulting from mandatory contributions to the pension plan. When pension plans are under-funded, companies sponsoring these plans are typically required by law to make additional financial contributions to close the funding gap. In the midst of a financial crisis, sponsoring companies often have limited borrowing capacity. Additional contributions to pension plans can worsen the financial constraint even further. Chapter 4 shows that the company's optimal investment strategy should depend on the amount of capital available for investment and on the initial pension funding ratio. The amount of capital available for investment is the sum of the internal capital and the amount of capital borrowed less pension contributions. Firms with lower pension funding ratio should have a lower investment threshold value than otherwise identical firms with better funded pension plans. The investment threshold value is the lowest project value above which the firm will invest. The result is driven by the fact that lower pension funding ratios mean higher expected future pension contributions and therefore lower values of waiting. The risk that in the future the capital available for investment will be used to fill the pension funding gap decreases the value of waiting. In other words, firms facing low pension plan funding ratios invest more aggressively than otherwise identical firms whose pension funds are better funded, unless they are extremely constrained or unconstrained. Chapter 4 find that the value of the investment project can increase dramatically in response to adopting our proposed investment strategy as compared to the optimal strategy ignoring the pension liabilities.

Chapter 2 Pension Fund Portfolio Wealth under Short-Term Regulation

This Chapter is based on Shi and Werker (2009a).

I Introduction

This chapter investigates the economic consequences of a difference in the planning horizon between an institutional investor pursuing long-term investment strategies and a regulator enforcing solvency of the institutional investor on a short-term basis. Such misalignment of horizons between an institutional investor and a regulator are likely to exist in most developed financial markets and affect, for example, banks, insurance companies, and pension funds. These investors are usually constrained in optimizing their respective objective function in the sense that they have to fulfill restrictions imposed by the financial markets authorities.

Consider the case of a bank operating under the regulatory rules of the Basel Committee on Bank Supervision. According to the 1996 Amendment the bank is constrained to hold market risk capital in the magnitude of a multiple of the 10-day value-at-risk (VaR) for the revaluation loss of the bank's trading book. According to Basel II the bank will be required to hold credit risk capital determined by 1-year default probabilities and expected shortfalls. These regulatory horizons are likely to stand in sharp contrast with the horizon of long-term investment projects involving payoffs in a more distant future the bank has to evaluate with respect to their creditworthiness.

A second case, to which we will refer again throughout this chapter, is a pension fund, which faces long-term pension liabilities with typical maturities of 15 years or more under a regulatory framework which imposes short-term solvency constraints. A recent example can be observed in the Netherlands where a pension regulatory regime (Financieel Toetsings Kader, FTK) is effective as from January 2007. According to the Dutch regulation, the pension funds should always keep the probability of underfunding one year ahead below 2.5%. Other countries that adopt value-at-risk in their pension fund regulation include Mexico and Australia.

The existence of such funding constraints can be understood in light of the recent experience of a simultaneous decrease in pension assets due to a poor stock market performance and an increase in pension liabilities due to low interest rates. For the UK, the Bank of England estimates the aggregate funding deficit of the FTSE-100 companies reaches GBP 57 billion, or 5.7% of their aggregate market capitalization, at 31 October 2003 (BNP Paribas 2004) while for the Netherlands the average funding ratio dropped from 130% in 2000 to 101% in 2002 (Ponds and van Riel 2007). The situation in the US is equally alarming. The funding deficit in America's corporate pension funds is estimated to be 350bn USD (Jørgensen 2007).

In this chapter we attempt to quantify the possible economic costs regulatory constraints create for the institutional investor. The examples above demonstrate the particular importance of VaR constraints in regulation practice despite the theoretical shortcomings of this risk measure (see Artzner et al.1999). For this reason we focus on VaR constraints imposed by the regulator. We study the institutional investor's optimal portfolio wealth when the regulatory horizon is as long as the investment horizon and when the regulatory horizon is shorter than the investment horizon. In the latter case, within the investor's investment horizon, there are a number of subsequent and non-overlapping regulatory checks and the investment horizon is divided into a few equal-length subperiods. In general, the investor has to insure his portfolio against the bad performance of the financial market to guarantee that (1) the current period VaR constraint is met and (2) there is enough wealth to fulfill the next periods VaR constraints. To do so, the investor has to hold more risk-free assets, thus, his ability to benefit from favorable financial market performance is limited. We show that more frequent regulatory checks generate higher costs. The costs increase less for the more risk-averse investor.

This chapter is related to the literature studying the optimal portfolio trading strategy under constraints. Grossman and Vila (1992) provide explicit solutions to optimal portfolio problems containing leverage and minimum portfolio return constraints. Basak (1995) and Grossman and Zhou (1995) focus on the impact of a specific VaR constraint, the portfolio insurance[1], on asset price dynamics in a general equilibrium model. Van Binsbergen and Brandt (2006) assess the influence of ex ante (preventive) and ex post (punitive) risk constraints on dynamic portfolio trading strategies. Ex ante risk constraints include, among others, VaR and short sell constraints. Ex post risk constraints include the loss of the investment manager's personal compensation and reputation when the portfolio wealth turns out to be low. They found that ex ante risk constraints tend to decrease gain from dynamic investment while ex post risk constraints can be welfare improving. We show that short-term VaR constraints, which allow a small probability that the portfolio wealth falls below the threshold value, can have a similar impact on the

[1] Portfolio insurance is a special case of VaR constraint, which requires the probability that the portfolio wealth falls below a certain threshold value to be zero.

portfolio wealth as portfolio insurance constraints, which require 100% probability that the portfolio wealth is above the threshold value.

Basak and Shapiro (2001) discuss the impact of the value-at-risk type regulation on the institutional investors' portfolio strategy. Their results show that a VaR constraint keeps the portfolio value above or at the threshold value, e.g. the value of the liability, when the investment environment (states of the world) is favorable but leaves his portfolio completely uninsured in the worst states of the world. The uninsured states of world are the worst states with probability of occurring equals to α. The probability is set by the regulator. The explanation is as follows. The VaR constrained investor is only concerned about the probability but not the magnitude of the loss. Therefore, the investor is willing to incur losses in compliance with the VaR constraint and it is optimal for him to incur losses in the states against which it is most expensive to insure. In Basak and Shapiro (2001) the VaR horizon is as long as the investment horizon. We extend the Basak and Shapiro (2001) paper by embedding subsequent and non-overlapping short-term value-at-risk type regulations in the portfolio optimization problem. We show that more frequent regulation can limit this moral hazard behavior due to the minimum amount of portfolio wealth required to fulfill future VaR constraints.

Cuoco et al. (2008) considers the optimal trading strategy of institutional investors under short-horizon VaR constraints assuming that the portfolio allocation over the VaR horizon is constant. We extend Cuoco et al. (2008) by allowing for optimal and time-varying portfolio allocation over the VaR horizon. This enables us to evaluate the cost of VaR regulation given that the institutional investor behaves optimally.

This chapter is also related to the literature about dynamic trading strategies of pension funds. Sundaresan and Zapatero (1997) considers an optimal asset allocation with a power utility function in final surplus. Boulier et al. (2005) assume a constant investment opportunity set with a risky and a risk-free asset. In their paper, the pension plan sponsor aims to minimize the expected discounted value of future contributions over a given horizon. Inkmann and Blake (2008) proposes a new approach to the valuation of pension obligations taking into account the asset allocation strategy and the underfunding risk of a pension fund. This chapter focuses on the optimal portfolio wealth of a pension fund when the regulatory horizon is shorter than its investment horizon and evaluates the economic cost of such regulation. Advantages of having frequent short-term VaR constraints include, among others, smaller expected portfolio wealth losses. A complete risk-return trade-off analysis is in our research agenda.

The outline of the chapter is as follows. Section 2 describes our model. Section 3 studies the optimal portfolio wealth under VaR constraints and section 4 discusses the economic costs of short-term value-at-risk type of regulation. Section 5 concludes.

II Financial Market And Investor Setup

We consider a continuous-time stochastic economy on a finite horizon $[0, T]$ in a complete financial market. A pension fund is considered to be a typical long-term institutional investor, maximizing expected utility of its funding ratio. There are two assets in the financial market, one is a riskless bond (cash account) and the other one is a risky stock. The price of the riskless asset evolves as

$$dB_t = rB_t dt, \text{ with } B_0 = 1, \tag{1}$$

where r denotes the constant risk-free rate. The price of stock, S_t, follows the diffusion process,

$$dS_t = (r + \lambda\sigma) S_t dt + \sigma S_t dw_t, \text{ with } S_0 = 1, \tag{2}$$

where w_t is a standard Brownian motion, λ is the price of risk and σ is the stock volatility. Dynamic market completeness (under no arbitrage) implies the existence of a unique state price density process, ζ, given by

$$d\zeta_t = -\zeta_t [rdt + \lambda dw_t].$$

As we assume an exogenously specified stream of liabilities[2] and a flat term structure, maximizing expected utility over the final funding ratio is equivalent to maximizing expected utility of final wealth. The pension fund invests a fraction π_t of his wealth in the risky stock. The pension fund's wealth, W_t, follows the dynamics

$$dW_t = W_t (r + \pi_t (\mu - r)) dt + \pi_t \sigma W_t dw_t. \tag{3}$$

We will assume a power utility function with constant relative risk aversion (CRRA) parameter γ. Finally, we abstract from new entitlements in the pension fund which implies that the pension fund's liabilities at time t are simply given as $L_t = L_0 \exp(rt)$.

[2]Liabilities become endogenous when they depend on the asset allocation. This holds, for example, when liabilities are inflation indexed conditional on the pension fund's funding ratio (see De Jong 2008, and Koijen and Nijman 2006) or when liabilities are calculated by discounting future pension payments with a default-adjusted discount rate as in Inkmann and Blake (2008).

No Regulatory Constraints

When no funding ratio constraints are imposed, the pension fund's optimization problem is,

$$\max_{W_T} E_0 \frac{W_T^{1-\gamma}}{1-\gamma} \tag{4}$$

$$s.t. \quad E_0 \left(\zeta_T W_T\right) \leq \zeta_0 W_0. \tag{5}$$

The solution to this problem is classical, but we provide a short recollection for expository reasons. Following the standard Martingale method by Cox and Huang (1989), the time-T optimal wealth of the pension fund is,

$$\begin{aligned} W_T^u &= \left(\phi_u \zeta_T\right)^{-\frac{1}{\gamma}}, \\ &= W_0 e^{-AT} \zeta_T^{-\frac{1}{\gamma}}, \end{aligned}$$

where W_T^u stands for the wealth of the pension fund at time T without regulation. The Lagrange Multiplier ϕ_u solves the budget constraint $\zeta_0 W_0 - E_0 \left(\zeta_T W_T^u\right) = 0$ and equals $W_0^{-\gamma} e^{AT\gamma}$ using the constant $A = r\left(1-\gamma\right)/\gamma + \frac{1}{2}\left(1-\gamma\right)/\gamma^2 \lambda^2$.

Without the VaR constraint, the value function at time 0 is

$$\begin{aligned} J_0^u \left(W_0\right) &= E_0 \left(\frac{\left(W_T^u\right)^{1-\gamma}}{1-\gamma}\right) \\ &= \frac{W_0^{1-\gamma}}{1-\gamma} \exp\left(AT\gamma\right). \end{aligned}$$

With Regulatory Constraints

The supervisor imposes a VaR type constraint on the pension fund: the probability that the funding ratio at time $t+\tau$ falls below one should not be larger than α, where α is usually a small number in the interval $[0,1]$. This can be formulated as

$$P_t \left(W_{t+\tau} < L_{t+\tau}\right) \leq \alpha, \quad t \in \left[0, T\right],$$

where τ, $\tau > 0$, is the regulatory horizon set by the regulator, $\alpha \in [0,1]$.

In the single-constraint model, the regulatory horizon τ is as long as the investment horizon. At time 0, the regulator requires that the probability of being under-funded at time T should be smaller than α, say 2.5%,

$$P_0 \left(W_T < L_T\right) \leq \alpha.$$

In the two-constraint and the more general multi-constraint models, the investment horizon stays the same but the regulatory horizons become shorter and shorter. For example, in the two-constraint model, the regulatory horizon could equal half of the investment horizon, which would imply the two VaR type constraints,

$$\begin{aligned} P_0\left(W_{T/2} < L_{T/2}\right) &\leq \alpha, \\ P_{T/2}\left(W_T < L_T\right) &\leq \alpha. \end{aligned}$$

In the following part of this section, we will describe these two models in details.

Single-Constraint Model

In our single-constraint model, the regulatory horizon coincides with the investment horizon, i.e., $\tau = T$. The optimization problem of the pension fund manager becomes

$$\begin{aligned} & \max_W E_0 \frac{W_T^{1-\gamma}}{1-\gamma} & \\ s.t. \qquad & E_0\left(\zeta_T W_T\right) \leq \zeta_0 W_0 & (6) \\ & P_0\left(W_T < L_T\right) \leq \alpha. & (7) \end{aligned}$$

The optimal wealth at time T depends on whether the VaR constraint, $P_0\left(W_T < L_T\right) \leq \alpha$, is binding or not. Since the optimal wealth without the VaR constraint, W_T^u, increases when W_0 increases, the VaR constraint becomes less binding for the pension fund with larger W_0. Because $\log \frac{\zeta_T}{\zeta_0}$ is normally distributed with mean $-\left(r + \frac{1}{2}\lambda^2\right)T$ and variance λT, it can be verified that when $W_0 \geq \theta_0$, where

$$\theta_0 = L_T \exp\left(-N^{-1}\left(\alpha\right)\frac{\lambda\sqrt{T}}{\gamma} - rT\frac{1}{\gamma} - \frac{1}{2}\lambda^2 T\frac{1}{\gamma} + AT\right),$$

the VaR constraint is not binding. In this case, the optimal wealth at time T is the same W_T^u as in the unconstrained case. Please see appendix A for the derivation of θ_0.

When the VaR constraint is binding, i.e.,$W_0 < \theta_0$, Basak and Shapiro (2001) prove that the optimal wealth at time T is

$$W_T^{(1)} = \begin{cases} \left(y^{(1)}\zeta_T\right)^{-\frac{1}{\gamma}} & \text{if } \zeta_T < \underline{\zeta}^{(1)} \\ L_T & \text{if } \underline{\zeta}^{(1)} \leq \zeta_T < \overline{\zeta}^{(1)} \\ \left(y^{(1)}\zeta_T\right)^{-\frac{1}{\gamma}} & \text{if } \zeta_T \geq \overline{\zeta}^{(1)} \end{cases} \qquad (8)$$

where $W_T^{(1)}$ stands for the wealth of the pension fund at time-T under long-term regulation, $\underline{\zeta}^{(1)} \equiv L_T^{-\gamma}/y^{(1)}$, $\overline{\zeta}^{(1)}$ is such that $P_0\left(\zeta_T > \overline{\zeta}^{(1)}\right) \equiv \alpha$, and $y^{(1)} \geq 0$ solves the budget constraint $\zeta_0 W_0 - E_0\left(\zeta_T W_T^{(1)}\right) = 0$.

We study the optimal portfolio wealth under a continuous CRRA utility function and a VaR constraint. The optimal portfolio wealth under a kinked utility function, for example, a loss-aversion utility function, is very similar to the one we obtained here. The portfolio insurance regulation provides a simple example. The problem with a continuous CRRA utility function and a portfolio insurance constraint is,

$$\begin{aligned} & \max_W E_0 \frac{W_T^{1-\gamma}}{1-\gamma} \\ s.t. \quad & E_0\left(\zeta_T W_T\right) \leq \zeta_0 W_0 \qquad (9) \\ & P_0\left(W_T < L_T\right) \leq 0, \qquad (10) \end{aligned}$$

The problem with a loss aversion utility function is,

$$\begin{aligned} & \max_W E_0 U\left(W_T\right), \\ s.t. \quad & E_0\left(\zeta_T W_T\right) \leq \zeta_0 W_0, \qquad (11) \end{aligned}$$

where

$$U\left(W_T\right) = \begin{cases} \frac{W_T^{1-\gamma}}{1-\gamma} & W_T \geq L_T \\ -\infty & W_T < L_T \end{cases}.$$

The solutions of both problems are

$$W_T^* = \begin{cases} \left(y\zeta_T\right)^{-\frac{1}{\gamma}} & \text{if } \zeta_T < \underline{\zeta} \\ L_T & \text{if } \underline{\zeta} \leq \zeta_T \end{cases}. \qquad (12)$$

In the setting with a power utility function and a portfolio insurance constraint, the regulator does not allow the portfolio wealth to be below L_T. In the case with a loss-aversion utility function, the investor is extremely unhappy when the portfolio wealth fall below L_T. Therefore, the optimal portfolio wealth under these two different settings is the same.

The optimal portfolio allocation at time t, $\forall t \in [0, T]$, is derived as follows. The portfolio wealth at time t, $W_t^{(1)}$, is $W_t^{(1)} = E_t\left[\frac{\zeta_T}{\zeta_t} W_T^{(1)}\right]$, that is,

$$W_t^{(1)} = \frac{1}{\zeta_t} E_t\left(\zeta_T (y^{(1)}\zeta_T)^{-\frac{1}{\gamma}} 1_{\{\zeta_T \leq \underline{\zeta}^{(1)}\}} + L_T 1_{\{\underline{\zeta}^{(1)} \leq \zeta_T \leq \overline{\zeta}^{(1)}\}} + \zeta_T (y^{(1)}\zeta_T)^{-\frac{1}{\gamma}} 1_{\{\overline{\zeta}^{(1)} \leq \zeta_T\}}\right). \qquad (13)$$

If we allocate π_t to the stock index, the diffusion process of the portfolio wealth is (3). Applying Ito's lemma to (13), we get

$$dW_t^{(1)} = ...dt - \frac{dW^{(1)}}{d\zeta}\zeta\lambda dw_t, 0 < t < T.$$

The diffusion terms of (13) and (3) must equal, therefore

$$\pi_t^{(1)} = \left[-\frac{dW^{(1)}}{d\zeta}\zeta\lambda\right]\frac{1}{\sigma W_t}.$$

After some algebra, we get

$$\begin{aligned}
\pi_t^{(1)} &= \frac{\lambda e^{\Gamma_t}}{\gamma \left(y^{(1)}\right)^{\frac{1}{\gamma}} \zeta_t^{\frac{1}{\gamma}} \sigma W_t^{(1)}} \left\{ N\left(d_1\left(\underline{\zeta}^{(1)}\right)\right) + 1 - N\left(d_1\left(\overline{\zeta}^{(1)}\right)\right) \right. \\
&\quad + \frac{\gamma\phi\left(d_1\left(\underline{\zeta}^{(1)}\right)\right) - \gamma\phi\left(d_1\left(\underline{\zeta}^{(1)}\right)\right)}{\left(y^{(1)}\zeta_t\right)^{-\frac{1}{\gamma}} \lambda\sqrt{T-t}} \\
&\quad \left. + \frac{\gamma L_T e^{-r(T-t)-\Gamma_t}\phi\left(d_2\left(\overline{\zeta}^{(1)}\right)\right) - \gamma L_T e^{-r(T-t)-\tau_t}\phi\left(d_2\left(\underline{\zeta}^{(1)}\right)\right)}{\left(y^{(1)}\zeta_t\right)^{-\frac{1}{\gamma}} \lambda\sqrt{T-t}} \right\}, \ \forall t \in [0, T],
\end{aligned}$$

where

$$\begin{aligned}
\Gamma_t &= \left(1 - \frac{1}{\gamma}\right)\left(r - \frac{1}{2}\lambda^2\right)(T-t) + \frac{1}{2}\left(1 - \frac{1}{\gamma}\right)^2 \lambda^2 (T-t) \\
d_2(x) &= \frac{\ln\frac{x}{\zeta_t} + \left(r - \frac{1}{2}\lambda^2\right)(T-t)}{\lambda\sqrt{T-t}}, \ x = \underline{\zeta}^{(1)} \text{ or } \overline{\zeta}^{(1)} \\
d_1(x) &= d_2(x) + \frac{1}{\gamma}\lambda\sqrt{T-t}, \ x = \underline{\zeta}^{(1)} \text{ or } \overline{\zeta}^{(1)}.
\end{aligned}$$

Two- and Multi-Constraint Models

In the two-constraint model, the investor's optimization problem is

$$\begin{aligned}
& \max_W E_0 \frac{W_T^{1-\gamma}}{1-\gamma} \\
s.t. \quad & P_0\left(W_{T/2} < L_{T/2}\right) \le \alpha \\
& P_{T/2}\left(W_T < L_T\right) \le \alpha \\
& E_0 \zeta_T W_T \le \zeta_0 W_0.
\end{aligned} \tag{14}$$

Our model directly embeds two VaR type constraints. We are going to use the backward iterative solution procedure to find the solution of (14). The size of $W_{T/2}$ will affect the bindingness of the VaR constraint during the next period.. However, at time $T/2$, for any given $W_{T/2}$, we know all possible optimal portfolio wealth at time T under the next period VaR constraint. Therefore, there is a one-to-one relationship between the wealth at time $T/2$, $W_{T/2}$, and the value of the value function, $E_{T/2}\frac{\left(W_T^{(2)}\right)^{1-\gamma}}{1-\gamma}$ which is the expected utility over optimal portfolio wealth at time T. Thus, the dynamic programming method is valid despite the fact that the bindingness of the next period depends on the current wealth.

First, we solve the maximization problem in the second period, that is, $[T/2, T]$. The second period problem is very similar to the single-constraint model. We assume that at time T/2, the pension fund starts with wealth $W_{T/2}$. Following the same method as in the single-constraint model, we find the optimal wealth at time T, $W_T^{(2)}$, and the value function at time T/2 $J_{T/2}^{(2)}\left(W_{T/2}\right)$. Second, we solve the maximization problem in the first period, that is, $[0, T/2]$. The difference between the maximization problem in the second and first period is that in the second period, the objective function is $\max E_{T/2}\frac{W_T^{1-\gamma}}{1-\gamma}$ and in the first period the objective function is $\max E_0 J_{T/2}\left(W_{T/2}\right)$ which is a kinked function. In the following part of this subsection, we are going to show the two steps in more detail.

First of all, we solve the maximization in the second period, i.e., $[T/2, T]$. By the law of iterated expectation, (14) can be rewritten as

$$\max_{W_T} E_0\left[E_{T/2}\frac{W_T^{1-\gamma}}{1-\gamma}\right] = \max_{W_T} E_{T/2}\frac{W_T^{1-\gamma}}{1-\gamma}.$$

Therefore, the maximization problem in the second period is

$$\begin{aligned} & \max_{W_T} E_{T/2}\frac{W_T^{1-\gamma}}{1-\gamma} \\ s.t. \quad & E_{T/2}\left(\zeta_T W_T\right) \leq \zeta_{T/2} W_{T/2} \\ & P_{T/2}\left(W_T < L_T\right) \leq \alpha. \end{aligned} \tag{15}$$

The optimization problem (15) is similar to the single-constraint model. When the wealth at time T/2, $W_{T/2}$, is not smaller than the threshold value, $\theta_{T/2}$, the optimal portfolio wealth at time T, $W_T^{(2)}$, is

$$W_T^{(2)} = W_{T/2}\zeta_{T/2}^{1/\gamma} e^{-0.5AT}\zeta_T^{-1/\gamma}, \tag{16}$$

and

$$\theta_{T/2} = L_T \exp\left(-\frac{1}{\gamma}\phi(\alpha)\lambda\sqrt{T/2} - \frac{1}{\gamma}\left(r + \frac{1}{2}\lambda^2\right)(T/2) + A(T/2)\right).$$

When the wealth at time T/2 is smaller than the threshold value, the optimal portfolio wealth at time T, $W_T^{(2)}$, is

$$W_T^{(2)} = \begin{cases} \left(y_{T/2}^{(2)}\zeta_T\right)^{-\frac{1}{\gamma}} & \text{if } \zeta_T < \underline{\zeta}_T^{(2)} \\ L_T & \text{if } \underline{\zeta}_T^{(2)} < \zeta_T < \overline{\zeta}_T^{(2)} \\ \left(y_{T/2}^{(2)}\zeta_T\right)^{-\frac{1}{\gamma}} & \text{if } \zeta_T \geq \overline{\zeta}_T^{(2)} \end{cases}, \tag{17}$$

where $\underline{\zeta}_T^{(2)} \equiv L_T^{-\gamma}/y_{T/2}^{(2)}$, $\overline{\zeta}_T^{(2)}$ is such that $P_{T/2}\left(\zeta_T > \overline{\zeta}_T^{(2)}\right) = \alpha$, and the Lagrange multiplier, $y_{T/2}^{(2)} \geq 0$, solves $\zeta_{T/2}W_{T/2} - E_{T/2}\left(\zeta_T W_T^{(2)}\right) = 0$.

$$J_{T/2}^{(2)}\left(W_{T/2}\right) = \begin{cases} E_{T/2}\left(\frac{\left(W_T^{(2)}\right)^{1-\gamma}}{1-\gamma}\right) & W_{T/2} < \theta_{T/2} \\ \frac{W_{T/2}^{1-\gamma}}{1-\gamma}e^{0.5AT\gamma} & W_{T/2} \geq \theta_{T/2} \end{cases}, \tag{18}$$

where

$$\begin{aligned} E_{T/2}\frac{\left(W_T^{(2)}\right)^{1-\gamma}}{1-\gamma} &= \frac{1}{1-\gamma}\left(y_{T/2}^{(2)}\zeta_{T/2}\right)^{1-\frac{1}{\gamma}} \times e^{\Gamma_{T/2}} \times N\left(D_1\left(\underline{\zeta}_T^{(2)}\right)\right) \\ &+\frac{L_T^{1-\gamma}}{1-\gamma}\left(N\left(D_2\left(\overline{\zeta}_T^{(2)}\right)\right) - N\left(D_2\left(\underline{\zeta}_T^{(2)}\right)\right)\right) \\ &+\frac{1}{1-\gamma}\left(y_{T/2}^{(2)}\zeta_{T/2}\right)^{1-\frac{1}{\gamma}} \times e^{\Gamma_{T/2}} \times N(-D_1\left(\overline{\zeta}_T^{(2)}\right)), \end{aligned}$$

and

$$\begin{aligned} D_2(x) &= \frac{\ln\frac{x}{\zeta_{T/2}} + 0.5\left(r + \frac{1}{2}\lambda^2\right)T}{\lambda\sqrt{T/2}},\ x = \underline{\zeta}^{(2)} \text{ or } \overline{\zeta}^{(2)}, \\ D_1(x) &= d_2(x) - \left(1 - \frac{1}{\gamma}\right)\lambda\sqrt{0.5T}, x = \underline{\zeta}^{(2)} \text{ or } \overline{\zeta}^{(2)}, \\ \Gamma_{T/2} &= -0.5\left(r + \frac{1}{2}\lambda^2\right)T\left(1 - \frac{1}{\gamma}\right) + \frac{1}{4}\left(1 - \frac{1}{\gamma}\right)^2\lambda^2 T. \end{aligned}$$

The derivation of (18) is provided in Appendix B.

Second, we solve the optimization problem in the first period,

$$\max_{W_{T/2}} E_0 J_{T/2}^{(2)}\left(W_{T/2}\right) \tag{19}$$

$$s.t. \quad \zeta_0 W_0 \geq E_0\left(\zeta_{T/2} W_{T/2}\right) \tag{20}$$

$$P_0\left(W_{T/2} < L_{T/2}\right) = \alpha \tag{21}$$

$$P_0\left(W_{T/2} < \underline{W_{T/2}}\right) = 0, \tag{22}$$

where $\underline{W_{T/2}}$ is the minimal portfolio wealth required to fulfill the next period's VaR constraint.

The maximization problem in the first-period has one extra constraint (22). We set this constraint because the minimum W_T which fulfills the funding ratio constraint or the VaR constraint is

$$W_T = \begin{cases} L_T & \text{if } \zeta_T < \underline{\zeta}_T \\ L_T & \text{if } \underline{\zeta}_T \leq \zeta_T < \overline{\zeta}_T \\ \left(y\zeta_T\right)^{-\frac{1}{\gamma}} & \text{if } \zeta_T \geq \overline{\zeta}_T \end{cases},$$

and, therefore, the minimum wealth at time T/2, $\underline{W_{T/2}}$, should equal to $\frac{1}{\zeta_{T/2}} E_{T/2} L_T \zeta_T 1_{\zeta_T < \overline{\zeta}_T}$. $+$ $\frac{1}{\zeta_{T/2}} E_{T/2}\left(y\zeta_T\right)^{-\frac{1}{\gamma}} \zeta_T 1_{\zeta_T > \overline{\zeta}_T}$. If the wealth at time T/2 is smaller than $\underline{W_{T/2}}$, it is not possible to fulfill the VaR constraint in the next period.

The Lagrange for the constrained optimization problem (19) is given by

$$\begin{aligned} L &= y_0^{(2)} \zeta_0 W_0 - y_2\left(1-\alpha\right) \\ &\quad + E_0\left[J_{T/2}^{(2)}\left(W_{T/2}\right) - y_0^{(2)} \zeta_{T/2} W_{T/2} + y_2 1_{W_{T/2} \geq L_{T/2}} - y_3 1_{W_{T/2} < \underline{W_{T/2}}}\right], \end{aligned} \tag{23}$$

where $y_0^{(2)}$, y_2 and y_3 are Lagrange multipliers solving $\zeta_0 W_0 - E_0\left(\zeta_{T/2} W_{T/2}\right) = 0$, $E_0 1_{W_{T/2} \geq L_{T/2}} = 1-\alpha$ and $E_0 1_{W_{T/2} < \underline{W_{T/2}}} - 0 = 0$, respectively, with $y_0^{(2)} \geq 0$, $y_2 \geq 0$ and $y_3 \geq 0$.

The first two terms of (23) are constants. Thus, finding a $W_{T/2}^{(2)}$ which maximizes the value of (23) is equivalent to finding a $W_{T/2}^{*}$ which maximizes the value of the function within $E_0\left[\cdot\right]$ in (23).

We let U_1 denote the part of value function above $\theta_{T/2}$ and U_2 denote the part below $\theta_{T/2}$. Let $\widetilde{W}_{1,T/2}$ denote the optimal wealth for the utility function U_1, $\widetilde{W}_{2,T/2}$ denote the optimal wealth for the utility function U_2. $\widetilde{W}_{1,T/2}$ and $\widetilde{W}_{2,T/2}$ have to satisfy the constraints $\widetilde{W}_{1,T/2} \geq \theta_{T/2}$ and $\widetilde{W}_{2,T/2} \leq \theta_{T/2}$ respectively. We compare these two local maxima pointwise to determine the global maximum, $W_{T/2}^{(2)}$. For all the $\zeta_{T/2}' s$ where

$L\left(\widetilde{W}_{1,T/2}\right) > (\leq) L\left(\widetilde{W}_{2,T/2}\right)$, the optimal wealth at time T/2, $W^{(2)}_{T/2}$, is $\widetilde{W}_{1,T/2}$ $\left(\widetilde{W}_{2,T/2}\right)$ where

$$\begin{aligned} L &= y_0^{(2)}\zeta_0 W_0 - y_2(1-\alpha) \\ &\quad + E_0\left[J^{(2)}_{T/2}\left(W_{T/2}\right) - y_0^{(2)}\zeta_{T/2}W_{T/2} + y_2 1_{W_{T/2}\geq L_{T/2}} - y_3 1_{W_{T/2}<\underline{W_{T/2}}}\right], \end{aligned} \tag{24}$$

and

$$J^{(2)}_{T/2}\left(W_{T/2}\right) = \begin{cases} U_1\left(W_{T/2}\right) & \text{if } W_{T/2} \geq \theta_{T/2} \\ U_2\left(W_{T/2}\right) & \text{if } W_{T/2} \leq \theta_{T/2} \end{cases}.$$

Let $I_{1j}(\cdot)$ be the inverse function of $U_j'(\cdot)$ where "$'$" is an abbreviation of the first order derivative. $\underline{\zeta}^{(2)}_{T/2,j}$, $\overline{\zeta}^{(2)}_{T/2,j}$, $\widehat{\zeta}^{(2)}_{T/2,j}$ and $\widetilde{\zeta}^{(2)}_{T/2,j}$ are defined as $I_j\left(y_0^{(2)}\underline{\zeta}^{(2)}_{T/2,j}\right) \equiv L_{T/2}$, $P_0\left(\zeta_{T/2} > \overline{\zeta}^{(2)}_{T/2,j}\right) \equiv \alpha$, $I_j\left(y_0^{(2)}\widehat{\zeta}^{(2)}_{T/2,j}\right) = \underline{W_{T/2}}$, and $I_j\left(y_0^{(2)}\widetilde{\zeta}^{(2)}_{T/2,j}\right) \equiv \theta_{T/2}$, for $j=1$ and 2, respectively. The portfolio wealth governed by U_j, $\widetilde{W}_{j,T/2}$, depends on the values of the state price deflator at time T/2, $\zeta_{T/2,}$ the floor of the portfolio wealth, $I_j\left(y_0^{(2)}\underline{\zeta}^{(2)}_{T/2,j}\right)$, the VaR probability, α, the minimum portfolio wealth at time T/2, $I_j\left(y_0^{(2)}\widehat{\zeta}^{(2)}_{T/2,j}\right)$, and the portfolio wealth above which the next period VaR constraint is not binding, $I_j\left(y_0^{(2)}\widetilde{\zeta}^{(2)}_{T/2,j}\right)$,

$$\widetilde{W}_{1,T/2} = \begin{cases} I_1\left(y_0^{(2)}\zeta_{T/2}\right)\times 1_{\zeta_{T/2}<\widetilde{\zeta}^{(2)}_{T/2,1}} + \theta_{T/2}\times 1_{\zeta_{T/2}\geq\widetilde{\zeta}^{(2)}_{T/2,1}}, & \text{if } \widetilde{\zeta}^{(2)}_{T/2,1} \leq \underline{\zeta}^{(2)}_{T/2,1}, \\ I_1\left(y_0^{(2)}\zeta_{T/2}\right)\times 1_{\zeta_{T/2}<\underline{\zeta}^{(2)}_{T/2,1}} + L_{T/2}\times 1_{\underline{\zeta}^{(2)}_{T/2,1}\leq\zeta_{T/2}<\overline{\zeta}^{(2)}_{T/2,1}} \\ \quad +\theta_{T/2}\times 1_{\zeta_{T/2}\geq\overline{\zeta}^{(2)}_{T/2,1}}, & \text{if } \overline{\zeta}^{(2)}_{T/2} \geq \widetilde{\zeta}^{(2)}_{T/2} > \underline{\zeta}^{(2)}_{T/2}, \\ I_1\left(y_0^{(2)}\zeta_{T/2}\right)\times 1_{\zeta_{T/2}<\underline{\zeta}^{(2)}_{T/2,1}} + L_{T/2}\times 1_{\underline{\zeta}^{(2)}_{T/2,1}\leq\zeta_{T/2}<\overline{\zeta}^{(2)}_{T/2,1}} \\ +I_1\left(y_0^{(2)}\zeta_{T/2}\right)\times 1_{\widetilde{\zeta}^{(2)}_{T/2,1}>\zeta_{T/2}\geq\overline{\zeta}^{(2)}_{T/2,1}} + \theta_{T/2}\times 1_{\zeta_{T/2}\geq\widetilde{\zeta}^{(2)}_{T/2,1}}, & \text{if } \widetilde{\zeta}^{(2)}_{T/2} > \overline{\zeta}^{(2)}_{T/2}, \end{cases}$$

and

$$\widetilde{W}_{2,T/2} = \begin{cases} \begin{array}{l} \theta_{T/2} \times 1_{\zeta_{T/2} < \widetilde{\zeta}^{(2)}_{T/2,2}}, \\ +I_2\left(y_0^{(2)}\zeta_{T/2}\right) \times 1_{\underline{\zeta}^{(2)}_{T/2,2} > \zeta_{T/2} \geq \widetilde{\zeta}^{(2)}_{T/2,2}} \\ +L_{T/2} \times 1_{\underline{\zeta}_{T/2,2} \leq \zeta_{T/2} < \overline{\zeta}_{T/2,2}} + \underline{W} \times 1_{\zeta_{T/2} \geq \overline{\zeta}_{T/2,2}}, \end{array} & \begin{array}{l} \text{if } \widetilde{\zeta}^{(2)}_{T/2,2} \leq \underline{\zeta}^{(2)}_{T/2,2} \\ \text{and } \widehat{\zeta}^{(2)}_{T/2,2} < \overline{\zeta}^{(2)}_{T/2,2}, \end{array} \\ \\ \begin{array}{l} \theta_{T/2} \times 1_{\zeta_{T/2} < \widetilde{\zeta}^{(2)}_{T/2,2}} + I_1\left(y_0^{(2)}\zeta_{T/2}\right) \\ \times 1_{\underline{\zeta}^{(2)}_{T/2,2} > \zeta_{T/2} \geq \widetilde{\zeta}^{(2)}_{T/2,2}} + L_{T/2} \times 1_{\underline{\zeta}^{(2)}_{T/2,2} \leq \zeta_{T/2} < \overline{\zeta}^{(2)}_{T/2,2}} \\ +I_2\left(y_0^{(2)}\zeta_{T/2}\right) \times 1_{\widehat{\zeta}^{(2)}_{T/2,2} > \zeta_{T/2} \geq \overline{\zeta}^{(2)}_{T/2,2}} \\ +\underline{W} \times 1_{\zeta_{T/2} \geq \widehat{\zeta}^{(2)}_{T/2,2}}, \end{array} & \begin{array}{l} \text{if } \widetilde{\zeta}_{T/2,2} \leq \underline{\zeta}_{T/2,2} \\ \text{and } \widehat{\zeta}_{T/2,2} \geq \overline{\zeta}_{T/2,2}, \end{array} \end{cases}$$

where the Lagrange multiplier, $y_0^{(2)}$, solves the equation $\zeta_0 W_0 - E_0\left(\zeta_{T/2} W_{T/2}\right) = 0$. For all the $\zeta'_{T/2}s$, the optimal portfolio wealth at time T/2, $W^{(2)}_{T/2}$, is $\widetilde{W}_{1,T/2}(\widetilde{W}_{2,T/2})$ if $\widetilde{W}_{1,T/2}(\widetilde{W}_{2,T/2})$ gives higher objective value. The derivation of $\widetilde{W}_{1,T/2}$, which is very similar to that of $\widetilde{W}_{2,T/2}$, is provided in Appendix C. The inverse of $U_2'(\cdot)$ is done numerically.

For any time i, where $0 \leq i \leq T/2$, the weight of the stock index in the whole portfolio, π_i, is

$$\pi_i^{(2)} = \left[-\frac{dW_i}{d\zeta}\zeta\lambda\right]\frac{1}{\sigma W_i},$$

where

$$\zeta_i W_i = E_i \zeta_{T/2} W^{(2)}_{T/2}.$$

$dW_i/d\zeta$ will be evaluated numerically.

We assume that the duration of the pension liability is 15 years and the VaR horizon is 1 year. Therefore, over a pension fund's 15-year investment horizon, there will be 15 non-overlapping VaR constraints. The value function at the end of each year is proxied with polynomials of the portfolio wealth to reduce computational time. The other steps to find the optimal portfolio wealth at the end of each year is similar to ones used to solve the optimal portfolio wealth at time T/2 in the two-constraint model.

III Optimal Portfolio Wealth and Economic Cost

In this section, we analyze the portfolio wealth and the economic cost of single-, two- and fifteen VaR constraints. We assume that $r = 0.04$, $\alpha = 2.5\%$, $\lambda = 0.2$, $L_0 = 0.9$, and $T = 15$. We will estimate the economic cost of VaR regulations with different regulatory horizons using the certainty equivalent loss.

The certainty equivalent loss, ce, measures the equivalent amount of wealth lost due to the VaR regulation and is defined as follows

$$J_0^*(W_0 - ce) = J_0^{VaR}(W_0),$$

where $J_0^*(\cdot)$ stands for the value function at time 0 without the VaR constraint, $J_0^*(W_0 - ce) = \frac{(W_0 - ce)^{1-\gamma}}{1-\gamma}\exp(AT\gamma)$, and $J_0^{VaR}(W_0)$ is the value function with the VaR constraint.

Portfolio Wealth and Portfolio Allocation

In the single-constraint model, the portfolio wealth at time T, $W_T^{(1)}$, is shown in (8). Figure 1 gives an example of $W_T^{(1)}$. The state price deflator, ζ_T, takes different values in different states of the world at time T. We use ω to indicate the states of the world, where $\omega \in \Omega$, and Ω refers to the sample space. As can be seen from (8) and figure 1, the regulated pension fund's optimal terminal wealth falls into three distinct regions in which the pension fund exhibits different investment behavior. In 'good' states ($\zeta_{T,\omega} \leq \underline{\zeta}$), the fund behaves as if there is no VaR constraint. In the 'intermediary region' ($\overline{\zeta} > \zeta_{T,\omega} > \underline{\zeta}$), the pension fund insures himself against loss. In the 'bad' region ($\zeta_{T,\omega} \geq \overline{\zeta}$), the fund is completely uninsured and incurs all the losses. The probability that the investor will end up in a 'bad' state is α.

Without the VaR constraint the optimal wealth at time T is $(y\zeta_T)^{-\frac{1}{\gamma}}$ which is a decreasing function of ζ_T. Recall that $\underline{\zeta}$ is defined as $\left(y\underline{\zeta}\right)^{-\frac{1}{\gamma}} \equiv L_T$. In the 'good' region where $\zeta_{T,\omega} \leq \underline{\zeta}$, the regulated pension fund can behave like an unregulated one because $\left(y\zeta_{T,\omega}\right)^{-\frac{1}{\gamma}} \geq L_T$. But for regions where $\zeta_{T,\omega} > \underline{\zeta}$, it holds that $\left(y\zeta_{T,\omega}\right)^{-\frac{1}{\gamma}} < L_T$[3]. Because the regulator requires that the maximum probability of under-funding should set to be α, the pension fund manager must make sure that at time T the value of the portfolio equals L_T in some of the states where $\zeta_{T,\omega} > \underline{\zeta}$. Recall that the quantity $\zeta_{T,\omega}$ can also be interpreted as the marginal cost at time 0 of obtaining one additional unit of wealth in state ω at time T. Hence, the larger the value of $\zeta_{T,\omega}$ the more expensive it is to insure. In order to limit the cost of insurance, the pension fund manager chooses to insure against the states $(\underline{\zeta}, \overline{\zeta}]$,where $P_0\left(\zeta_T > \overline{\zeta}\right) \equiv \alpha$, and leaves the worst states uninsured.

Figure 2 shows the optimal portfolio allocation at time T/4 with a VaR constraint in the one-constraint model. The VaR investor divides the values of the state price

[3] When the VaR constraint is binding, the probability that ζ_T is larger than $\underline{\zeta}$ is larger than α, that is, $P_0\left(\zeta_T > \underline{\zeta}\right) > \alpha$.

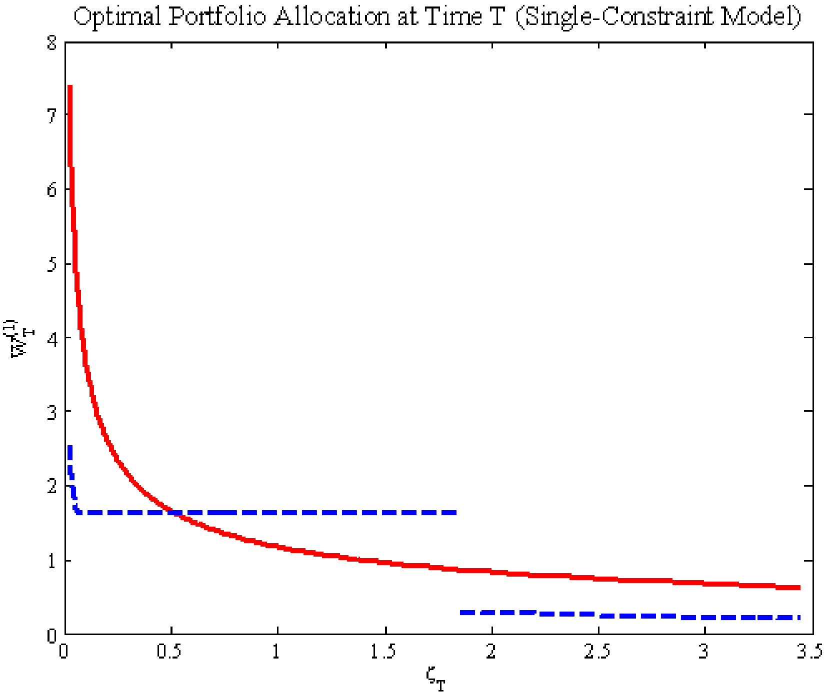


Figure 1: This figure shows the optimal portfolio wealth at time T following the VaR constraint in the single-constraint model. The parameter values are $r = 0.04$, $\alpha = 2.5\%$, $\lambda = 0.2$, $L_T = 0.9exp(rT)$, $\gamma = 2$, $W_0 = 0.81$ and $T = 15$. The solid line represents the optimal portfolio wealth without the VaR constraint. The dashed line represents the optimal portfolio wealth with the VaR constraint.

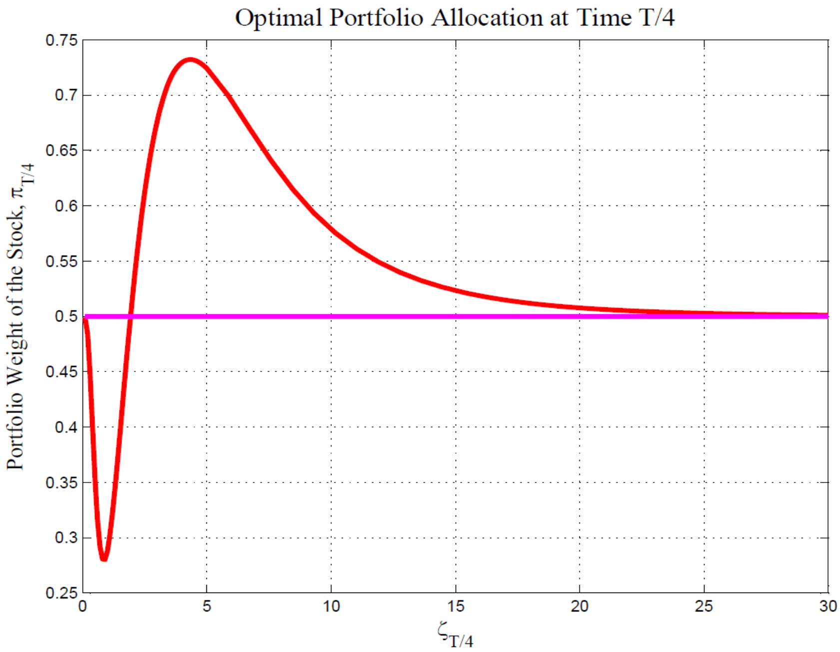


Figure 2: This figure shows the optimal portfolio allocation at time T/4 following a VaR constraint in the single-constraint model. The parameter values are $r = 0.04$, $\alpha = 2.5\%$, $\lambda = 0.2$, $L_T = 0.9exp(rT)$, $\gamma = 2$, $W_0 = 0.91$ and $T = 15$. The pink horizontal solid line represents the optimal portfolio allocation at time T/4 without the VaR constraint. The red curve represents the optimal portfolio allocation with the VaR constraint (One-Constraint Model).

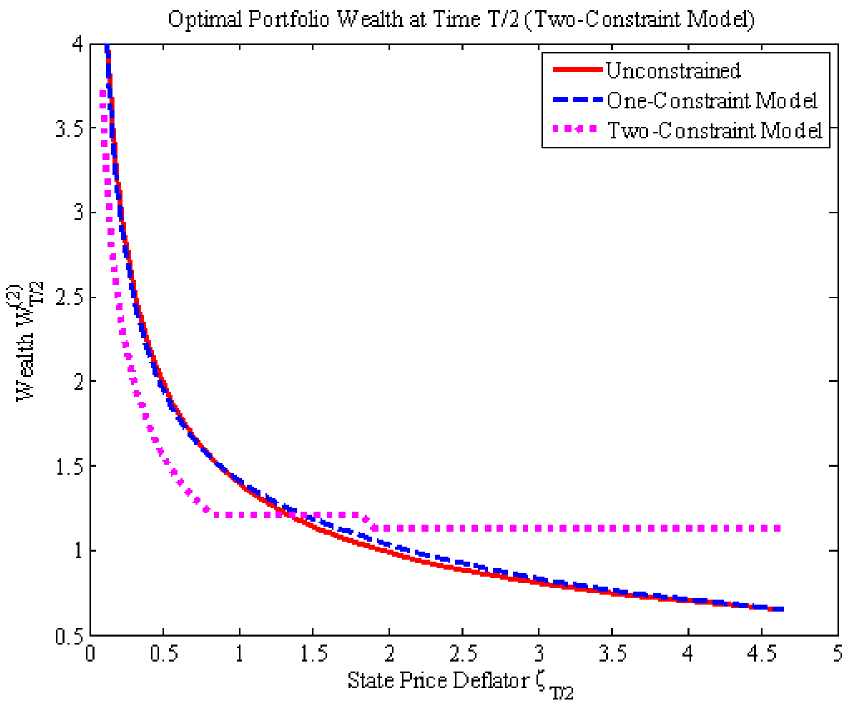


Figure 3: This figure shows the optimal portfolio wealth at time T/2 following VaR constraints in the two-constraint model. The parameter values are $r = 0.04$, $\alpha = 2.5\%$, $\lambda = 0.2$, $L_T = 0.9exp(rT)$, $L_{T/2} = 0.9exp(0.5rT)$, $\gamma = 2$, $W_0 = 1$, and $T = 15$. The red solid line represents the optimal portfolio wealth at time T/2 without VaR constraint. The dotted line represents the optimal portfolio wealth at time T/2 with two VaR constraints (Two-Constraint Model) and the dashed line represents the optimal portfolio wealth at time T/2 with one VaR constraint (Single-Constraint Model).

deflator into three intervals. The first interval is for $\zeta_{T/4} \leq 1$, the second interval is for $1 < \zeta_{T/4} \leq 5.5$, and the third interval is for $\zeta_{T/4} > 5.5$. At time 0, the chances that the VaR investor ends up in interval 1 at time $T/4$ is 72%, in interval 2 is 28%, and in interval 3 is almost 0. Within interval 1, as the investment environment becomes worse, the VaR investor allocates less to the stock index to fulfill VaR constraint at time T. Within interval 2, the VaR investor starts to increase allocation to the stock index. At that time, he is gambling because there is still a chance that he might end up in the "intermediate state" at time T where the VaR constraint is binding. In interval 3, the VaR investor behaves as if there is no VaR constraint.

In the two-constraint model, the portfolio wealth at time T has similar pattern as the one in the single-constraint model. However, the pattern of the portfolio wealth in the earlier period is different. This is because at the end of the first period, the pension fund has to make sure that there is enough money to fulfill the VaR constraint in the next period. Figure 3 shows the wealth at time T/2 in the two-constraint model, single-constraint model and the unconstrained model. We have derived the final wealth in the one period model in (8). The wealth in the single-constraint model before time T is derived in appendix D. As we can see from figure 3, in the 'good' states (low $\zeta_{T/2}$) the two-constraint model generates lower wealth than the single-constraint model and the unconstrained wealth. In the 'intermediate' states, the portfolio wealth of the two-constraint model becomes higher than that of the single-constraint model and the unconstrained model. In the 'bad' states (high $\zeta_{T/2}$), the portfolio wealth of the two-constraint model is always above about 1.12. This result does not hold for the single- period and unconstrained models. The fact that the portfolio wealth of the two-constraint model is always above 1.12 is because it is not possible to fulfill the VaR constraint in the second period (from time T/2 to time T) if the portfolio wealth at time T/2 is smaller than 1.12. The minimum wealth needed to fulfill future VaR constraint depends on future liability values, the VaR probability, α, and discounter factor. For example, if α changes to 0.05, the minimum wealth at time T/2 is 1.07 and if r changes to 5%, the minimum wealth is 1.22. α, pension liabilities and discount factor are assumed to be exogenous, therefore, the minimum wealth is also exogenous. If the minimum wealth is higher (lower) than 1.12, the portfolio wealth at 'good' states will be smaller (larger) than exhibited in figure 3. In order to have better performance in the 'bad' states, the pension fund with two VaR constraints has to invest more in risk-less assets. Holding more risk-less assets reduces his ability to benefit from stock price increases.

As the VaR horizon becomes shorter, the minimal portfolio wealth required to fulfill all future VaR constraints increases. Eventually, when the VaR horizon is shortened to be 1 year (15-constraint model), the minimal portfolio wealth is almost as high as the pension liability each year (before the 14th year). For example, figure 4 shows the optimal portfolio allocation at the end of year 13, where the investment horizon is 15 years, γ is 2, and the regulatory horizon is 1 year. That is, at the beginning of each year before year 14, the pension fund manager will have to make sure that the portfolio wealth at the end of year has to be as high as the pension liability. Thus, a number of short-term non-overlapping VaR constraints can have the same impact on the portfolio wealth as the portfolio insurance strategy where the probability that the pension portfolio wealth falls below the pension liability is required to be 0.

Figure 5 shows the portfolio allocation at time T/4 in one-constraint, two-constraint and fifteen-constraint model. When the investment environment is favorable ($\zeta_{T/4} \leq 1$), both the two-constraint VaR investor and fifteen-constraint VaR investor reduce their allocation to stock to meet their VaR constraints. When $1 < \zeta_{T/4} \leq 2$, the two-constraint

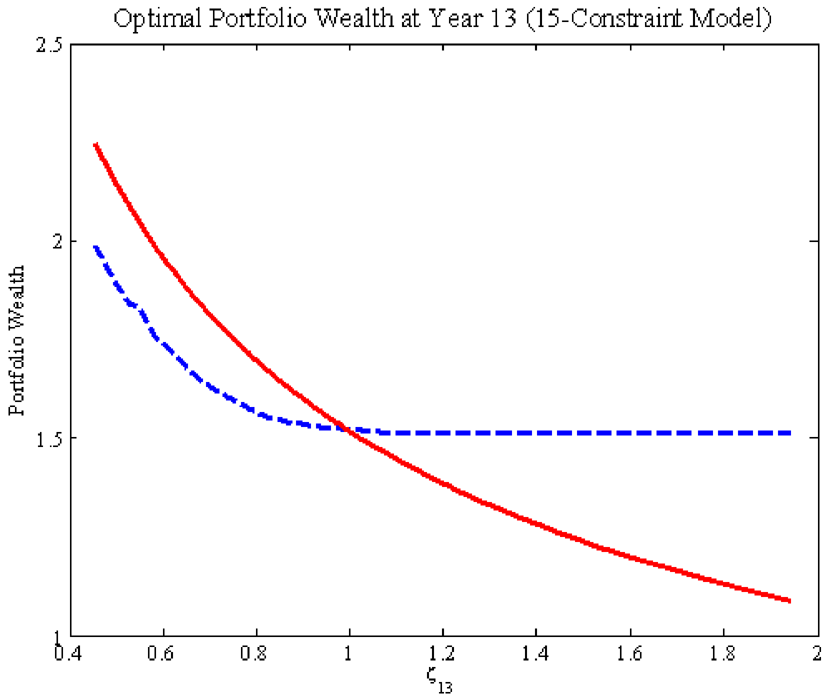


Figure 4: This figure shows the optimal portfolio wealth at year 13 following VaR constraints in the fifteen-constraint model. The parameter values are $r = 0.04$, $\alpha = 2.5\%$, $\lambda = 0.2$, $L_T = 0.9exp(rT)$, $L_{T/2} = 0.9exp(0.5rT)$, $\gamma = 2$, $W_{12} = 1.48$, and $T = 15$. The VaR horizon is one year. The red solid line represents the optimal portfolio wealth at year 13 without VaR constraint. The dashed line represents the optimal portfolio wealth at year 13 with 15 VaR constraints (fifteen-Constraint Model).

VaR investor increases his allocation to stock. This is the gambling behavior explained before. When $\zeta_{T/4} > 2$ (at time 0, the probability that $\zeta_{T/4} > 2$ is very close 0), the two-constraint investor decreases his allocation to the stock index. Eventually, the allocation to the stock index converges to 0. The fifteen-constraint VaR investor exhibits gambling behavior at much smaller $\zeta'_{T/4}s$ since for his VaR constraint is much more binding than other investors.

Certainty Equivalent Losses

In this sub-section, we will assess the economic cost having a more frequent VaR regulation. Figure 6 shows the certainty equivalent losses of single-constraint, two-constraint and fifteen-constraint models when $\gamma = 2$. Figure 7 shows the certainty equivalent losses for $\gamma = 4$. From figure 6 and figure 7, we can draw three main conclusions. First, the certainty equivalent loss is increasing as the initial portfolio wealth decreases. For example, for a pension fund with risk aversion of 2 and initial wealth of 0.91, the certainty equivalent loss is 0.0341 for the single-constraint model. The certainty equivalent loss decreases to 0.0069 as the initial wealth increases to 1.1. Second, the certainty equivalent loss increases with the regulatory frequency. For a pension fund with $\gamma = 2$ and 100% funded (i.e., initial wealth equals 0.91), the certainty equivalent loss is 0.0341 for the single-constraint model where the investment horizon is as long as the regulatory horizon, 0.0460 for the two-constraint model where the regulatory horizon is half as long as the investment horizon and 0.1005 for the fifteen-constraint model where the pension fund faces a VaR constraint every year. That is, for a pension fund with $\gamma = 2$, the cost of having annual VaR constraint equals to 11% (0.1005/0.91) reduction in its current portfolio wealth. Third, the certainty equivalent is decreasing as the coefficient of relative risk aversion increases. For example, in the Fifteen-constraint model, if a pension fund has initial wealth of 0.91 the certainty equivalent loss is 0.1005 if the fund's risk aversion is 2 and 0.0464 if the fund's risk aversion is 4 which equals to 5.1% of reduction in its current asset value.

The certainty equivalent costs of a 15-constraint portfolio insurance regulation are very similar to the 15-constraint VaR regulation. For example, for a pension fund with gamma equals to 2 and initial wealth equals to 0.91, 0.95 and 1.0, the certainty equivalent losses are about 10%, 7% and 6% respectively. This is not surprising since the minimum portfolio wealth in the 15-constraint VaR model is almost as large as the liability values, as shown in figure 4. If an investor faces a continuous VaR constraint, at any point of time except time T, his minimum portfolio wealth required to fulfill future VaR constraints is higher than other investors. In the extreme case, if the investor has to invest all the time

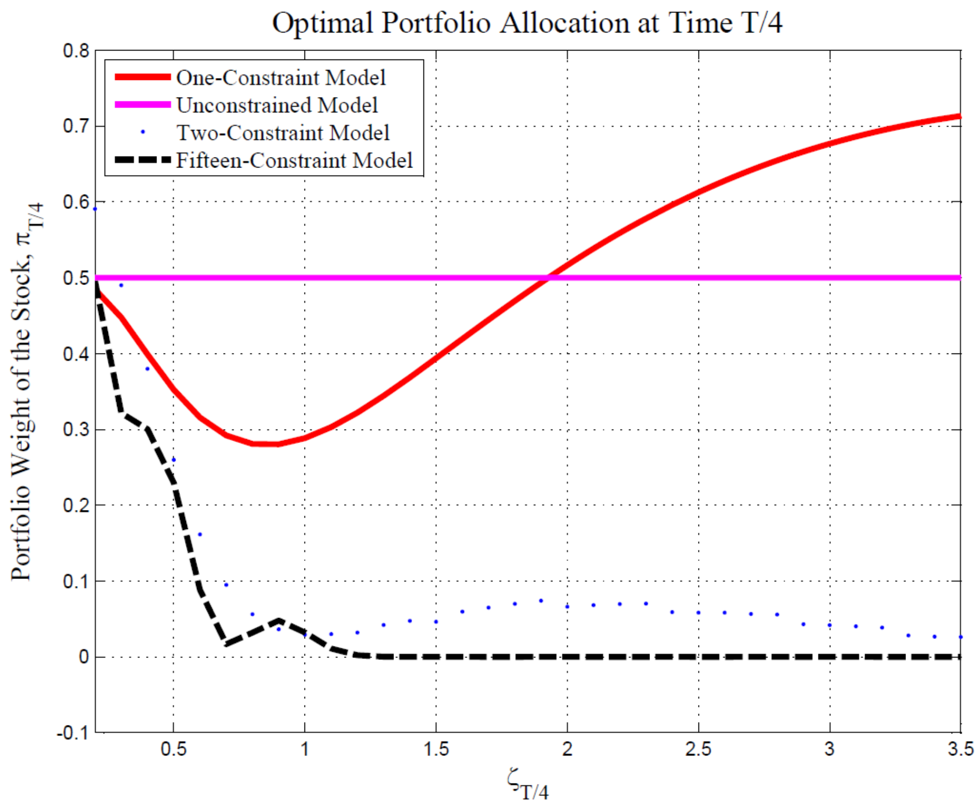


Figure 5: This figure shows the optimal portfolio allocation at time T/4 following VaR constraints in the one-constraint, two-constraint and fifteen-constraint model. The parameter values are $r = 0.04$, $\alpha = 2.5\%$, $\lambda = 0.2$, $L_T = 0.9exp(rT)$, $L_{T/2} = 0.9exp(0.5rT)$, $\gamma = 2$, $W_0 = 1$, and $T = 15$. The red solid line represents the optimal portfolio wealth at time T/4 without VaR constraint. The red curve represents portfolio wealth at time T/4 with one VaR constraint (One-Constraint Model). The dotted line represents the optimal portfolio wealth at time T/4 with two VaR constraints (Two -Constraint Model) and the dashed line represents the optimal portfolio wealth at time T/4 with 15 VaR constraints (Fifteen-Constraint Model).

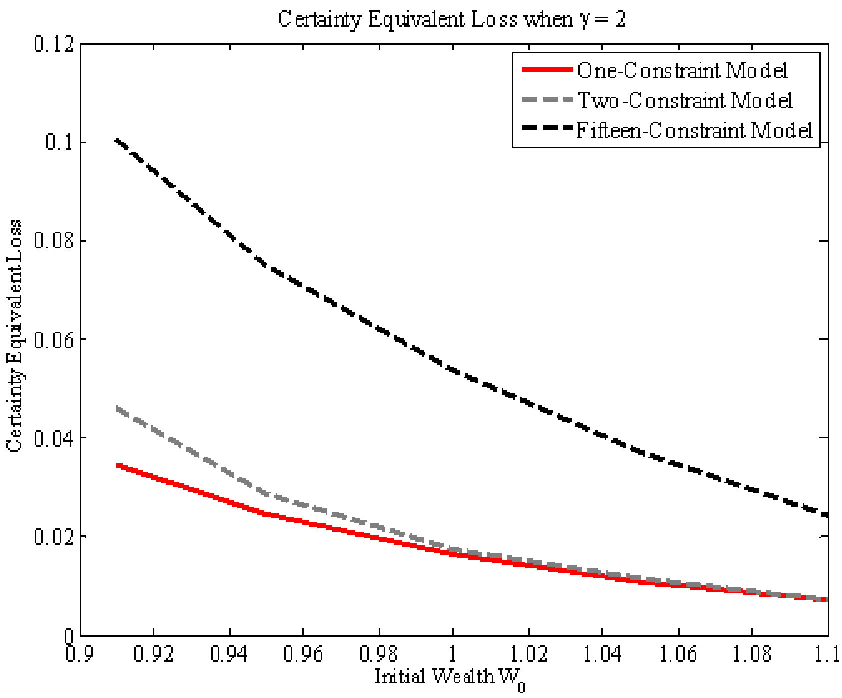


Figure 6: This figure shows the certainty equivalent loss following single, two and fifteen VaR constraints for $\gamma = 2$. The parameter values are $r = 0.04$, $\alpha = 2.5\%$, $\lambda = 0.2$, $L_T = 0.9exp(rT)$, $L_{T/2} = 0.9exp(0.5rT)$, and the investment horizon, T, is 15 years.

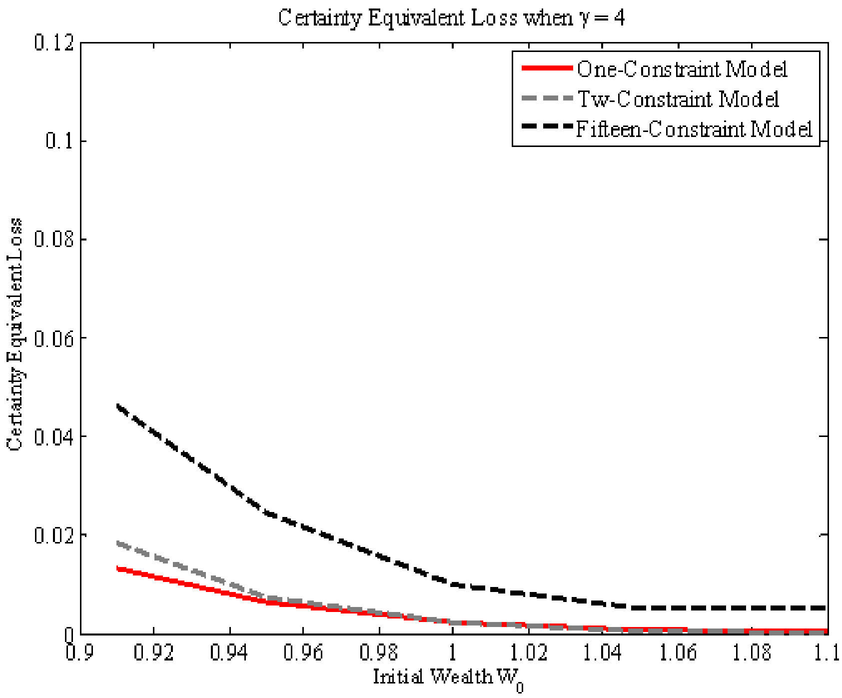


Figure 7: This figure shows the certainty equivalent loss following single, two and fifteen VaR constraints for $\gamma = 4$. The parameter values are $r = 0.04$, $\alpha = 2.5\%$, $\lambda = 0.2$, $L_T = 0.9exp(rT)$, $L_{T/2} = 0.9exp(0.5rT)$, and the investment horizon, T, is 15 years.

in bonds, the certainty equivalent loss is 14% of his initial wealth for an investor with $\gamma = 2$.

IV Conclusions

The value-at-risk type constraint is often adopted by regulators to limit the portfolio risk of institutional investors. However, the regulatory horizon is usually much shorter than the institutional investors' investment horizon. We study three models with different regulatory horizons. In our single-constraint model, the regulatory horizon is as long as the investment horizon. In our two-constraint model, the regulatory horizon is half as long as the investment horizon. Thus, the investor faces two subsequent and non-overlapping regulatory constraints within his investment horizon. In the fifteen-constraint model, the investor faces a VaR constraint every year over his 15-year investment horizon. We find that in the multi- constraint model the investor invests more in the risk-free asset than the single-constraint model. Shorter regulatory constraint, on one hand, enables the pension fund to avoid large losses when the investment environment worsens but, on the other hand, also limits the pension fund ability to benefit from an increase in stock prices. We show that the economic cost increases with the regulatory frequency but it increases less for the more risk-averse investor. For a 100% funded pension fund, the cost brought by the annual VaR constraint can be as large as 10% reduction in its current asset value. Pension funds could face instantaneous liquidation resulting from unfavorable financial performances. The liquidation of pension funds will push the world into a deeper recession. Therefore, even though short horizon VaR constraint is restrictive in portfolio allocation, but such a regulation is necessary since it reduces the discontinuity risk dramatically.

Appendix

A The Deviation of θ_0

Without the VaR constraint, the optimal wealth at time T is

$$W_T^u = W_0 e^{-AT} \zeta_T^{-\frac{1}{\gamma}},$$

where the state-price deflator ζ_T is expressed as

$$\zeta_T = \zeta_0 e^{-r(T-0)-\lambda(w_T-w_0)-\frac{1}{2}\lambda^2(T-0)},$$

with $\zeta_0 = 1$ and $w_T - w_0 \sim N(0, T-0)$. Assuming that $P_0(W_T^u < L_T) \leq \alpha$ holds, we have

$$\begin{aligned} P_0\left(W_T^u < L_T\right) &= P_0\left(W_0 e^{-AT} \zeta_T^{-\frac{1}{\gamma}} < L_T\right) \\ &= N\left(\frac{\gamma}{\lambda\sqrt{T-0}} \ln\left(\frac{L_T}{W_0} e^{-r(T-0)\frac{1}{\gamma} - \frac{1}{2}\lambda^2 (T-0)\frac{1}{\gamma} + AT}\right)\right) \\ &\leq \alpha. \end{aligned} \tag{A1}$$

The inequality arises because $w_T - w_0$ is normally distributed with mean 0 and variance $T-0$. From (A1), we can obtain the threshold value, θ_0, and

$$\theta_0 = L_T \exp\left(-N^{-1}(\alpha)\frac{\lambda\sqrt{T-0}}{\gamma} - r(T-0)\frac{1}{\gamma} - \frac{1}{2}\lambda^2 (T-0)\frac{1}{\gamma} + A(T-0)\right).$$

As soon as $W_0 \geq \theta_0$, the VaR constraint is not binding.

B The Derivation of $J_{T/2}^{(2)}\left(W_{T/2}\right)$

When the VaR constraint is binding, the optimal wealth at time T is shown in (16) and (17). The value function at time T/2, $J_{T/2}^{(2)}\left(W_{T/2}\right)$, is defined as $E_{T/2}\frac{\left(W_T^{(2)}\right)^{1-\gamma}}{1-\gamma}$. When the VaR constraint in the second period is not binding, the value function at time T/2 is

$$\begin{aligned} E_{T/2}\frac{\left(W_T^{(2)}\right)^{1-\gamma}}{1-\gamma} &= E_{T/2}\frac{\left(W_{T/2}\zeta_{T/2}^{1/\gamma} e^{-A(T-T/2)}\zeta_T^{-1/\gamma}\right)^{1-\gamma}}{1-\gamma} \\ &= \frac{W_{T/2}^{1-\gamma}}{1-\gamma} e^{A(T-T/2)\gamma}. \end{aligned}$$

When the VaR constraint in the second period is binding, the value function at time T/2 is

$$E_{T/2}\frac{\left(W_T^{(2)}\right)^{1-\gamma}}{1-\gamma} = E_{T/2}\left(\frac{\left(y^{(2)}\zeta_T\right)^{1-\frac{1}{\gamma}}}{1-\gamma} 1_{\left\{\zeta_T \leq \underline{\zeta}^{(2)}\right\}} + \frac{L_T^{1-\gamma}}{1-\gamma} 1_{\left\{\underline{\zeta}^{(2)} \leq \zeta_T \leq \overline{\zeta}^{(2)}\right\}} + \frac{\left(y^{(2)}\zeta_T\right)^{1-\frac{1}{\gamma}}}{1-\gamma} 1_{\left\{\overline{\zeta}^{(2)} \leq \zeta_T\right\}}\right). \tag{A2}$$

Let $Y_{T-T/2} \equiv -r(T-T/2) - \lambda\left(w_T - w_{T/2}\right) - \frac{1}{2}\lambda^2 (T-T/2)$, therefore

$$\zeta_T = \zeta_{T/2} e^{Y_{T-T/2}}$$

where $Y_{T-T/2}$ has the distribution$Y_{T-T/2} \sim N\left(\left(-r - \frac{1}{2}\lambda^2\right)(T-T/2), \lambda^2 (T-T/2)\right)$.

The inequality, $\zeta_T \leq \underline{\zeta}^{(2)}$, can be rewritten as

$$Y_{T-T/2} \leq \ln \frac{\underline{\zeta}^{(2)}}{\zeta_{T/2}},$$

The first term of (A2) can be derived as follows

$$\begin{aligned}
& \int_{-\infty}^{\ln \frac{\underline{\zeta}^{(2)}}{\zeta_{T/2}}} \frac{\left(y^{(2)}\zeta_T\right)^{1-\frac{1}{\gamma}}}{1-\gamma} \frac{1}{\sqrt{2\pi\lambda^2 (T-T/2)}} e^{-\frac{1}{2\lambda^2(T-T/2)}\left(Y_{T-T/2}+\left(r+\frac{1}{2}\lambda^2\right)(T-T/2)\right)^2} dY_{T-T/2} \\
= & \int_{-\infty}^{\frac{\ln \frac{\underline{\zeta}^{(2)}}{\zeta_{T/2}}+\left(r+\frac{1}{2}\lambda^2\right)(T-T/2)}{\lambda\sqrt{T-T/2}}} \frac{\left(y^{(2)}\zeta_T\right)^{1-\frac{1}{\gamma}}}{1-\gamma} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}Z^2} dZ \\
= & \frac{1}{1-\gamma}\left(y^{(2)}\zeta_{T/2}\right)^{1-\frac{1}{\gamma}} \int_{-\infty}^{D_2\left(\underline{\zeta}^{(2)}\right)} e^{\left(1-\frac{1}{\gamma}\right)Y_{T-T/2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}Z^2} dZ \\
= & \frac{1}{1-\gamma}\left(y\zeta_{T/2}\right)^{1-\frac{1}{\gamma}} \int_{-\infty}^{D_2\left(\underline{\zeta}^{2}\right)} e^{\left(1-\frac{1}{\gamma}\right)\left(\lambda\sqrt{T-T/2}Z-\left(r+\frac{1}{2}\lambda^2\right)(T-T/2)\right)} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}Z^2} dZ \\
= & \frac{1}{1-\gamma}\left(y^{(2)}\zeta_{T/2}\right)^{1-\frac{1}{\gamma}} \times e^{\Gamma_{T/2}} \times \int_{-\infty}^{D_2\left(\underline{\zeta}^{(2)}\right)} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(Z-\left(1-\frac{1}{\gamma}\right)\lambda\sqrt{T-T/2}\right)^2} dZ \\
= & \frac{1}{1-\gamma}\left(y^{(2)}\zeta_{T/2}\right)^{1-\frac{1}{\gamma}} \times e^{\Gamma_{T/2}} \times N(D_1\left(\underline{\zeta}^{(2)}\right))
\end{aligned}$$

where

$$\begin{aligned}
D_2(x) &= \frac{\ln \frac{x}{\zeta_{T/2}} + \left(r+\frac{1}{2}\lambda^2\right)(T-T/2)}{\lambda\sqrt{T-T/2}}, \ x = \underline{\zeta}^{(2)} \text{ or } \overline{\zeta}^{(2)}, \\
D_1(x) &= d_2(x) - \left(1-\frac{1}{\gamma}\right)\lambda\sqrt{T-T/2}, x = \underline{\zeta}^{(2)} \text{ or } \overline{\zeta}^{(2)}, \\
\Gamma_{T/2} &= -\left(r+\frac{1}{2}\lambda^2\right)(T-T/2)\left(1-\frac{1}{\gamma}\right) + \frac{1}{2}\left(1-\frac{1}{\gamma}\right)^2 \lambda^2 (T-T/2), \\
Z &= \frac{Y_{T-T/2} + \left(r+\frac{1}{2}\lambda^2\right)(T-T/2)}{\lambda\sqrt{T-T/2}}
\end{aligned}$$

With similar steps, we can also derive the second and third terms of $(A2)$.

C The Derivation of $\widetilde{W}_{1,T/2}$

The procedure to find $\widetilde{W}_{1,T/2}$ is as follows. $\widetilde{W}_{1,T/2}$ is the solution of

$$\widetilde{W}_{1,T/2} = \max_{W_{1,T/2}} \left\{ U_1\left(W_{T/2}\right) - y\zeta_{T/2}W_{T/2} + y_2 1_{W_{T/2}\geq L_{T/2}} - y_3 1_{W_{T/2}<\underline{W_{T/2}}} - y_4 1_{W_{T/2}\leq\theta_{T/2}} \right\},$$

where

$$y_{2,1}=\begin{cases} U_1\left(\theta_{T/2}\right)-y\bar{\zeta}^{(2)}_{T/2}\theta_{T/2}-U_1\left(L_{T/2}\right)+y\bar{\zeta}^{(2)}_{T/2}L_{T/2} & \text{if } \widetilde{\zeta}^{(2)}_{T/2}\leq\bar{\zeta}^{(2)}_{T/2} \\ U_1\left(I_1\left(y\bar{\zeta}^{(2)}_{T/2}\right)\right)-y\bar{\zeta}^{(2)}_{T/2}I_1\left(y\bar{\zeta}^{(2)}_{T/2}\right)-U_1\left(L_{T/2}\right)+y\bar{\zeta}^{(2)}_{T/2}L_{T/2} & \text{if } \widetilde{\zeta}^{(2)}_{T/2}>\bar{\zeta}^{(2)}_{T/2} \end{cases},$$

$y_3=\infty$, $y_4=\infty$ and $I_1(\cdot)$ is the inverse function of $U_1(\cdot)$. The function on which $max\{\cdot\}$ operates is not concave in $W_{T/2}$, but can only exhibit four local maxima at $I_1\left(y\zeta_{T/2}\right)$, $L_{T/2}$, $\theta_{T/2}$ and $\underline{W_{T/2}}$. Let $\mathcal{L}$ denote the function on which $max\{\cdot\}$ operates. Since $\theta_{T/2}$ is the amount of wealth above which the VaR constraint is not binding and $\underline{W_{T/2}}$ is the amount of wealth below which the VaR constraint can never be fulfilled, $\theta_{T/2}$ will not be smaller than $\underline{W_{T/2}}$. Therefore, in order to find the $\widetilde{W}_{1,T/2}$, we only have to compare the values of $\mathcal{L}\left(I_1\left(y\zeta_{T/2}\right)\right)$, $\mathcal{L}\left(L_{T/2}\right)$, and $\mathcal{L}\left(\theta_{T/2}\right)$ because $\widetilde{W}_{1,T/2}$ has to be larger than or equal to $\theta_{T/2}$ $\forall\zeta_{T/2}$.

Note that $\underline{\zeta}^{(2)}_{T/2,1}$, $\bar{\zeta}^{(2)}_{T/2,1}$, $\widehat{\zeta}^{(2)}_{T/2,1}$ and $\widetilde{\zeta}^{(2)}_{T/2,1}$ are defined as $I_1\left(y^{(2)}_0\underline{\zeta}^{(2)}_{T/2,1}\right)\equiv L_{T/2}$, $P_0\left(\zeta_{T/2}>\bar{\zeta}^{(2)}_{T/2,1}\right)\equiv\alpha$, $I_1\left(y^{(2)}_0\widehat{\zeta}^{(2)}_{T/2,1}\right)=\underline{W_{T/2}}$, and $I_1\left(y^{(2)}_0\widetilde{\zeta}^{(2)}_{T/2,j}\right)\equiv\theta_{T/2}$ respectively.

<u>Case 1</u>: $\theta_{T/2}\geq L_{T/2}$

When $\theta_{T/2}\geq L_{T/2}$, the possible portfolio values are $I_1\left(y\zeta_{T/2}\right)$ and $\theta_{T/2}$. For all $\zeta_{T/2}\leq\widetilde{\zeta}^{(2)}_{T/2}$, $I_1\left(y\zeta_{T/2}\right)\geq\theta_{T/2}$ because $I_1\left(y\zeta_{T/2}\right)$ is a strictly decreasing function of $\zeta_{T/2}$. And for all $\zeta_{T/2}>\widetilde{\zeta}^{(2)}_{T/2}$, $I_1\left(y\zeta_{T/2}\right)<\theta_{T/2}$. For all $\zeta'_{T/2}s$ which are not larger than $\widetilde{\zeta}^{(2)}_{T/2}$, the values of the function where $max\{\cdot\}$ operates on become

$$\begin{aligned}\mathcal{L}\left(I_1\left(y\zeta_{T/2}\right)\right)&=U_1\left(I_1\left(y\zeta_{T/2}\right)\right)-y\zeta_{T/2}I_1\left(y\zeta_{T/2}\right)+y_2,\\ \mathcal{L}\left(\theta_{T/2}\right)&=U_1\left(\theta_{T/2}\right)-y\zeta_{T/2}\theta_{T/2}+y_2.\end{aligned}$$

For simplicity we use y to represent $y^{(2)}_0$ throughout Appendix C. The first order derivative of $U_1\left(W_{T/2}\right)-y\zeta_{T/2}W_{T/2}$ with respect to $W_{T/2}$ evaluated at $W_{T/2}=\theta_{T/2}$ is $y\left(\widetilde{\zeta}^{(2)}_{T/2}-\zeta_{T/2}\right)$ and $y\left(\widetilde{\zeta}^{(2)}_{T/2}-\zeta_{T/2}\right)$ is larger than 0 for all $\zeta_{T/2}\leq\widetilde{\zeta}^{(2)}_{T/2}$. Therefore, when $\theta_{T/2}>L_{T/2}$ and $\zeta_{T/2}\leq\widetilde{\zeta}^{(2)}_{T/2}$, the optimal portfolio wealth governed by U_1 is $I_1\left(y\zeta_{T/2}\right)$. When $\theta_{T/2}>L_{T/2}$ and $\zeta_{T/2}>\widetilde{\zeta}^{(2)}_{T/2}$, we have $I_1\left(y\zeta_{T/2}\right)<\theta_{T/2}$ $\forall\zeta_{T/2}>\widetilde{\zeta}^{(2)}_{T/2}$. The values of the objective function are

$$\begin{aligned}\mathcal{L}\left(I_1\left(y\zeta_{T/2}\right)\right)&=U_1\left(I_1\left(y\zeta_{T/2}\right)\right)-y\zeta_{T/2}I_1\left(y\zeta_{T/2}\right)+y_2-\infty,\\ \mathcal{L}\left(\theta_{T/2}\right)&=U_1\left(\theta_{T/2}\right)-y\zeta_{T/2}\theta_{T/2}+y_2.\end{aligned}$$

It is obvious that $\mathcal{L}\left(I_1\left(y\zeta_{T/2}\right)\right)$ is smaller than $\mathcal{L}\left(L_{T/2}\right)$, the optimal portfolio wealth when $\theta_{T/2} > L_{T/2}$ and $\zeta_{T/2} > \widetilde{\zeta}^{(2)}_{T/2}$ is $\theta_{T/2}$.

Case 2: $\theta_{T/2} < L_{T/2}$

When $\theta_{T/2} < L_{T/2}$, there are three candidate portfolio wealth, $I_1\left(y\zeta_{T/2}\right)$, $L_{T/2}$ and $\theta_{T/2}$. When $\zeta_{T/2} < \underline{\zeta}^{(2)}_{T/2}$, we have $I_1\left(y\zeta_{T/2}\right) > L_{T/2}$, and $\mathcal{L}\left(I_1\left(y\zeta_{T/2}\right)\right) > \mathcal{L}\left(L_{T/2}\right) > \mathcal{L}\left(\theta_{T/2}\right)$, where

$$\begin{aligned}
\mathcal{L}\left(I_1\left(y\zeta_{T/2}\right)\right) &= U_1\left(I_1\left(y\zeta_{T/2}\right)\right) - y\zeta_{T/2}I_1\left(y\zeta_{T/2}\right) + y_2,\\
\mathcal{L}\left(L_{T/2}\right) &= U_1\left(L_{T/2}\right) - y\zeta_{T/2}L_{T/2} + y_2,\\
\mathcal{L}\left(\theta_{T/2}\right) &= U_1\left(\theta_{T/2}\right) - y\zeta_{T/2}\theta_{T/2}.
\end{aligned}$$

The inequality, $\mathcal{L}\left(I_1\left(y\zeta_{T/2}\right)\right) > \mathcal{L}\left(L_{T/2}\right) > \mathcal{L}\left(\theta_{T/2}\right)$ follows from the fact that the first order derivatives of $\left\{U_1\left(W_{T/2}\right) - y\zeta_{T/2}W_{T/2}\right\}$ with respect to $W_{T/2}$ evaluated at $L_{T/2}$ and $\theta_{T/2}$ are larger than 0 and $y_2 \geq 0$. So the optimal portfolio wealth governed by U_1 is, $\widetilde{W}_{1,T/2} = I_1\left(y\zeta_{T/2}\right)$, for $\zeta_{T/2} < \underline{\zeta}^{(2)}_{T/2}$.

To find the optimal portfolio wealth when $\underline{\zeta}^{(2)}_{T/2} \leq \zeta_{T/2} < \overline{\zeta}^{(2)}_{T/2}$, we have to distinguish two sub-cases, those are, $\widetilde{\zeta}^{(2)}_{T/2} \leq \overline{\zeta}^{(2)}_{T/2}$ and $\widetilde{\zeta}^{(2)}_{T/2} > \overline{\zeta}^{(2)}_{T/2}$. When $\widetilde{\zeta}^{(2)}_{T/2} \leq \overline{\zeta}^{(2)}_{T/2}$ and $\underline{\zeta}^{(2)}_{T/2} \leq \zeta_{T/2} < \widetilde{\zeta}^{(2)}_{T/2}$, we have $y_2 = U_1\left(\theta_{T/2}\right) - y\overline{\zeta}^{(2)}_{T/2}\theta_{T/2} - U_1\left(L_{T/2}\right) + y\overline{\zeta}^{(2)}_{T/2}L_{T/2}$, $\theta_{T/2} < I_1\left(y\zeta_{T/2}\right) < L_{T/2}$, and

$$\begin{aligned}
\mathcal{L}\left(I_1\left(y\zeta_{T/2}\right)\right) &= U_1\left(I_1\left(y\zeta_{T/2}\right)\right) - y\zeta_{T/2}I_1\left(y\zeta_{T/2}\right),\\
\mathcal{L}\left(L_{T/2}\right) &= U_1\left(L_{T/2}\right) - y\zeta_{T/2}L_{T/2} + y_2\\
&= U_1\left(\theta_{T/2}\right) - y\zeta_{T/2}L_{T/2} + y\overline{\zeta}^{(2)}_{T/2}\left(L_{T/2} - \theta_{T/2}\right)\\
\mathcal{L}\left(\theta_{T/2}\right) &= U_1\left(\theta_{T/2}\right) - y\zeta_{T/2}\theta_{T/2}.
\end{aligned}$$

Since $L_{T/2} > \theta_{T/2}$, we have $d\left\{y\zeta_{T/2}\left(L_{T/2} - \theta_{T/2}\right)\right\}/d\zeta_{T/2} = y\left(L_{T/2} - \theta_{T/2}\right) > 0$ and

$$\begin{aligned}
&\mathcal{L}\left(L_{T/2}\right) - \mathcal{L}\left(\theta_{T/2}\right)\\
=\;& U_1\left(\theta_{T/2}\right) + y\overline{\zeta}^{(2)}_{T/2}\left(L_{T/2} - \theta_{T/2}\right) - \left\{U_1\left(\theta_{T/2}\right) - y\zeta_{T/2}\left(L_{T/2} - \theta_{T/2}\right)\right\}\\
>\;& 0.
\end{aligned}$$

For $\zeta_{T/2} > \underline{\zeta}^{(2)}_{T/2}$, it holds that $d\left[U_1\left(I_1\left(y\zeta_{T/2}\right)\right) + y_1\zeta_{T/2}\left(L_{T/2} - I_1\left(y\zeta_{T/2}\right)\right)\right]/d\zeta_{T/2} =$

$y\left(L_{T/2}-I_1\left(y\zeta_{T/2}\right)\right)>0$. Therefore, for $\underline{\zeta}^{(2)}_{T/2}\leq\zeta_{T/2}<\widetilde{\zeta}^{(2)}_{T/2}$,

$$\begin{aligned}
& \mathcal{L}\left(L_{T/2}\right)-\mathcal{L}\left(I_1\left(y\zeta_{T/2}\right)\right) \\
= \; & U_1\left(\theta_{T/2}\right)+y\overline{\zeta}^{(2)}_{T/2}\left(L_{T/2}-\theta_{T/2}\right)-\left\{U_1\left(I_1\left(y\zeta_{T/2}\right)\right)-y\zeta_{T/2}\left[L_{T/2}-I_1\left(y\zeta_{T/2}\right)\right]\right\} \\
> \; & 0.
\end{aligned}$$

Thus, when $\widetilde{\zeta}^{(2)}_{T/2}\leq\overline{\zeta}^{(2)}_{T/2}$ and $\underline{\zeta}^{(2)}_{T/2}\leq\zeta_{T/2}<\widetilde{\zeta}^{(2)}_{T/2}$, the optimal portfolio wealth is $L_{T/2}$.

When $\widetilde{\zeta}^{(2)}_{T/2}\leq\overline{\zeta}^{(2)}_{T/2}$ and $\widetilde{\zeta}^{(2)}_{T/2}\leq\zeta_{T/2}<\overline{\zeta}^{(2)}_{T/2}$, we have $y_2=U_1\left(\theta_{T/2}\right)-y\overline{\zeta}^{(2)}_{T/2}\theta_{T/2}-U_1\left(L_{T/2}\right)+y\overline{\zeta}^{(2)}_{T/2}L_{T/2}$, $I_1\left(y\zeta_{T/2}\right)<\theta_{T/2}<L_{T/2}$, and

$$\begin{aligned}
\mathcal{L}\left(I_1\left(y\zeta_{T/2}\right)\right) &= U_1\left(I_1\left(y\zeta_{T/2}\right)\right)-y\zeta_{T/2}I_1\left(y\zeta_{T/2}\right)-y_{3,1} \\
&= U_1\left(I_1\left(y\zeta_{T/2}\right)\right)-y\zeta_{T/2}I_1\left(y\zeta_{T/2}\right)-\infty, \\
\mathcal{L}\left(L_{T/2}\right) &= U_1\left(\theta_{T/2}\right)-y\zeta_{T/2}L_{T/2}+y\overline{\zeta}^{(2)}_{T/2}\left(L_{T/2}-\theta_{T/2}\right), \\
\mathcal{L}\left(\theta_{T/2}\right) &= U_1\left(\theta_{T/2}\right)-y\zeta_{T/2}\theta_{T/2}.
\end{aligned}$$

Since $d\left\{y\zeta_{T/2}\left(L_{T/2}-\theta_{T/2}\right)\right\}/d\zeta_{T/2} = y\left(L_{T/2}-\theta_{T/2}\right) > 0$, therefore, $LM\left(L_{T/2}\right) > LM\left(\theta_{T/2}\right)$, and the optimal portfolio wealth is $L_{T/2}$. To summarize, when $\widehat{\zeta}_{T/2}\leq\overline{\zeta}^{(2)}_{T/2}$ and $\underline{\zeta}_{T/2}\leq\zeta_{T/2}<\overline{\zeta}^{(2)}_{T/2}$, the optimal portfolio wealth is $L_{T/2}$.

When $\widetilde{\zeta}^{(2)}_{T/2}>\overline{\zeta}^{(2)}_{T/2}$ and $\underline{\zeta}^{(2)}_{T/2}\leq\zeta_{T/2}<\overline{\zeta}^{(2)}_{T/2}$, $y_2=U_1\left(I_1\left(y\overline{\zeta}^{(2)}_{T/2}\right)\right)-y\overline{\zeta}^{(2)}_{T/2}I_1\left(y\overline{\zeta}^{(2)}_{T/2}\right)-U_1\left(L_{T/2}\right)+y\overline{\zeta}^{(2)}_{T/2}L_{T/2}$, we have $\theta_{T/2}<I_1\left(y\zeta_{T/2}\right)<L_{T/2}$, and

$$\begin{aligned}
\mathcal{L}\left(I_1\left(y\zeta_{T/2}\right)\right) &= U_1\left(I_1\left(y\zeta_{T/2}\right)\right)-y_1\zeta_{T/2}I_1\left(y\zeta_{T/2}\right), \\
\mathcal{L}\left(L_{T/2}\right) &= U_1\left(L_{T/2}\right)-y_1\zeta_{T/2}L_{T/2}+y_{2,1} \\
&= U_1\left(I_1\left(y\overline{\zeta}^{(2)}_{T/2}\right)\right)-y_1\zeta_{T/2}L_{T/2}+y\overline{\zeta}^{(2)}_{T/2}\left(L_{T/2}-I_1\left(y\overline{\zeta}^{(2)}_{T/2}\right)\right) \\
&\geq U_1\left(\theta_{T/2}\right)-y_1\zeta_{T/2}\theta_{T/2}+y\overline{\zeta}^{(2)}_{T/2}\left(L_{T/2}-I_1\left(y\overline{\zeta}^{(2)}_{T/2}\right)\right), \\
\mathcal{L}\left(\theta_{T/2}\right) &= U_1\left(\theta_{T/2}\right)-y_1\zeta_{T/2}\theta_{T/2}.
\end{aligned}$$

Since $L_{T/2}>I_1\left(y\zeta_{T/2}\right)$, it holds that $\mathcal{L}\left(L_{T/2}\right)>\mathcal{L}\left(\theta_{T/2}\right)$. For $\underline{\zeta}^{(2)}_{T/2}\leq\zeta_{T/2}<\overline{\zeta}^{(2)}_{T/2}$, we have $d\left\{U_1\left(I_1\left(y\zeta_{T/2}\right)\right)+y\zeta_{T/2}\left(L_{T/2}-I_1\left(y\zeta_{T/2}\right)\right)\right\}/d\zeta_{T/2}=y\left(L_{T/2}-I_1\left(y\zeta_{T/2}\right)\right)>0$, and thus,

$$
\begin{aligned}
& \mathcal{L}\left(L_{T/2}\right) - \mathcal{L}\left(I_1\left(y\zeta_{T/2}\right)\right) \\
= \; & U_1\left(I_1\left(y\overline{\zeta}^{(2)}_{T/2}\right)\right) + y\overline{\zeta}^{(2)}_{T/2}\left(L_{T/2} - I_1\left(y\overline{\zeta}^{(2)}_{T/2}\right)\right) \\
& - \left\{U_1\left(I_1\left(y\zeta_{T/2}\right)\right) - y_1\zeta_{T/2}\left[L_{T/2} - I_1\left(y\zeta_{T/2}\right)\right]\right\}.
\end{aligned}
$$

Therefore, when $\widehat{\zeta}_{T/2} > \overline{\zeta}^{(2)}_{T/2}$ and $\underline{\zeta}_{T/2} \leq \zeta_{T/2} < \overline{\zeta}^{(2)}_{T/2}$, the optimal portfolio wealth is $L_{T/2}$.

When $\widetilde{\zeta}^{(2)}_{T/2} \leq \overline{\zeta}^{(2)}_{T/2}$ and $\overline{\zeta}^{(2)}_{T/2} \leq \zeta_{T/2}$, it holds that $I_1\left(y\zeta_{T/2}\right) < \theta_{T/2}$, and

$$
\begin{aligned}
\mathcal{L}\left(I_1\left(y\zeta_{T/2}\right)\right) &= U_1\left(I\left(y\zeta_{T/2}\right)\right) - y\zeta_{T/2} I\left(y\zeta_{T/2}\right) - \infty \\
\mathcal{L}\left(L_{T/2}\right) &= U_1\left(L_{T/2}\right) - y\zeta_{T/2} L_{T/2} + y_2 \\
&= U_1\left(\theta_{T/2}\right) - y\zeta_{T/2} L_{T/2} + y\overline{\zeta}^{(2)}_{T/2}\left(L_{T/2} - \theta_{T/2}\right) \\
\mathcal{L}\left(\theta_{T/2}\right) &= U_1\left(\theta_{T/2}\right) - y\zeta_{T/2}\theta_{T/2}.
\end{aligned}
$$

$LM\left(L_{T/2}\right) - LM\left(\theta_{T/2}\right) = y\left(L_{T/2} - \theta_{T/2}\right)\left(\overline{\zeta}^{(2)}_{T/2} - \zeta_{T/2}\right)$ which is smaller than 0 for $\overline{\zeta}^{(2)}_{T/2} \leq \zeta_{T/2}$. Therefore, the optimal portfolio wealth is $\theta_{T/2}$.

When $\widetilde{\zeta}^{(2)}_{T/2} > \overline{\zeta}^{(2)}_{T/2}$ and $\overline{\zeta}^{(2)}_{T/2} \leq \zeta_{T/2} < \widetilde{\zeta}^{(2)}_{T/2}$, we have $I_1\left(y\zeta_{T/2}\right) > \theta_{T/2}$, and

$$
\begin{aligned}
\mathcal{L}\left(I_1\left(y\zeta_{T/2}\right)\right) &= U_1\left(I_1\left(y\zeta_{T/2}\right)\right) - y_1\zeta_{T/2} I_1\left(y\zeta_{T/2}\right), \\
\mathcal{L}\left(L_{T/2}\right) &= U_1\left(L_{T/2}\right) - y_1\zeta_{T/2} L_{T/2} + y_2 \\
&= U_1\left(I_1\left(y\overline{\zeta}^{(2)}_{T/2}\right)\right) - y_1\zeta_{T/2} L_{T/2} + y\overline{\zeta}^{(2)}_{T/2}\left(L_{T/2} - I_1\left(y\overline{\zeta}^{(2)}_{T/2}\right)\right) \\
&> U_1\left(\theta_{T/2}\right) - y_1\zeta_{T/2}\theta_{T/2} + y\overline{\zeta}^{(2)}_{T/2}\left(L_{T/2} - I_1\left(y\overline{\zeta}^{(2)}_{T/2}\right)\right), \\
\mathcal{L}\left(\theta_{T/2}\right) &= U_1\left(\theta_{T/2}\right) - y_1\zeta_{T/2}\theta_{T/2},
\end{aligned}
$$

Because $L_{T/2} > I_1\left(y\overline{\zeta}^{(2)}_{T/2}\right)$ and $d\left\{U_1\left(W_{T/2}\right) - y\zeta_{T/2} W_{T/2}\right\}/dW_{T/2} > 0$, we can get $\mathcal{L}\left(I_1\left(y\zeta_{T/2}\right)\right) - \mathcal{L}\left(\theta_{T/2}\right) > y\overline{\zeta}^{(2)}_{T/2}\left(L_{T/2} - I_1\left(y\overline{\zeta}^{(2)}_{T/2}\right)\right)$ and $\mathcal{L}\left(I_1\left(y\zeta_{T/2}\right)\right) - \mathcal{L}\left(L_{T/2}\right) > 0$. Thus, the optimal portfolio wealth is $I_1\left(y\zeta_{T/2}\right)$.

When $\widetilde{\zeta}^{(2)}_{T/2} > \overline{\zeta}^{(2)}_{T/2}$ and $\zeta_{T/2} \geq \widetilde{\zeta}^{(2)}_{T/2}$, $I_1\left(y\zeta_{T/2}\right) < \theta_{T/2}$,

$$\begin{aligned}
\mathcal{L}\left(I_1\left(y\zeta_{T/2}\right)\right) &= U_1\left(I_1\left(y\zeta_{T/2}\right)\right) - y\zeta_{T/2}I_1\left(y\zeta_{T/2}\right) - \infty, \\
\mathcal{L}\left(L_{T/2}\right) &= U_1\left(L_{T/2}\right) - y\zeta_{T/2}L_{T/2} + y_2 \\
&= U_1\left(I_1\left(y\overline{\zeta}^{(2)}_{T/2}\right)\right) - y\zeta_{T/2}L_{T/2} + y\overline{\zeta}^{(2)}_{T/2}\left(L_{T/2} - I_1\left(y\overline{\zeta}^{(2)}_{T/2}\right)\right), \\
\mathcal{L}\left(\theta_{T/2}\right) &= U_1\left(\theta_{T/2}\right) - y\zeta_{T/2}\theta_{T/2}.
\end{aligned}$$

Since $d\left\{U_1\left(I_1\left(y\zeta_{T/2}\right)\right) + y\zeta_{T/2}\left(L_{T/2} - I_1\left(y\zeta_{T/2}\right)\right)\right\}/d\zeta_{T/2} = y\left(L_{T/2} - I_1\left(y\zeta_{T/2}\right)\right)$ which is larger than 0 for $\zeta_{T/2} > \underline{\zeta}^{(2)}_{T/2}$, $\mathcal{L}\left(\theta_{T/2}\right) - \mathcal{L}\left(L_{T/2}\right) > 0$, therefore, the optimal portfolio wealth is $\theta_{T/2}$.

$U_2(\cdot)$ is also a convex function like $U_1(\cdot)$. The procedure to find $\widetilde{W}_{2,T/2}$ which is the optimal portfolio wealth governed by $U_2\left(W_{T/2}\right)$ is, therefore, the same as the one for $\widetilde{W}_{1,T/2}$.

Appendix D

The time-t $(t \leq T)$ optimal wealth is given by

$$\begin{aligned}
W^{VaR}(t) &= \frac{e^{\Gamma_t}}{(y\zeta_t)^{\frac{1}{\gamma}}} N\left(d_1\left(\underline{\zeta}\right)\right) + L_T e^{-r(T-t)}\left[N\left(-d_2\left(\underline{\zeta}\right)\right) - N\left(-d_2\left(\overline{\zeta}\right)\right)\right] \\
&\quad + \frac{e^{\Gamma_t}}{(y\zeta_t)^{\frac{1}{\gamma}}} N\left(-d_1\left(\overline{\zeta}\right)\right) \qquad \text{(A4)}
\end{aligned}$$

where

$$\begin{aligned}
\Gamma_t &= \left(1 - \frac{1}{\gamma}\right)\left(r - \frac{1}{2}\lambda^2\right)(T-t) + \frac{1}{2}\left(1 - \frac{1}{\gamma}\right)^2 \lambda^2 (T-t) \\
d_2(x) &= \frac{\ln\frac{x}{\zeta_t} + \left(r - \frac{1}{2}\lambda^2\right)(T-t)}{\lambda\sqrt{T-t}}, \ x = \underline{\zeta} \text{ or } \overline{\zeta} \\
d_1(x) &= d_2(x) + \frac{1}{\gamma}\lambda\sqrt{(T-t)}, \ x = \underline{\zeta} \text{ or } \overline{\zeta}
\end{aligned}$$

Proof. Because $\zeta_t W_t$ is a martingale, the wealth at time t can be written as

$$W^{VaR}(t) = E\left[\frac{\zeta_T}{\zeta_t}W^{(1)}_T | F_t\right]. \qquad \text{(A5)}$$

When r and λ are constant, conditional on the information set F_t, $\ln\zeta_T$ is normally distributed with mean $\ln\zeta_t - (r + \frac{1}{2}\lambda^2)(T-t)$ and variance $\lambda^2(T-t)$. Inserting (8) into

(A5) and evaluating the conditional expectations over each of the three regions of ζ_T yields

$$W^{VaR}(t) = \frac{1}{\zeta_t} E_t \left(\zeta_T (y\zeta_T)^{-\frac{1}{\gamma}} 1_{\{\zeta_T \leq \underline{\zeta}\}} + L_T 1_{\{\underline{\zeta} \leq \zeta_T \leq \bar{\zeta}\}} + \zeta_T (y\zeta_T)^{-\frac{1}{\gamma}} 1_{\{\bar{\zeta} \leq \zeta_T\}} \right).$$

First, we are going to derive the first term of $(A4)$, i.e., $\frac{e^{\Gamma t}}{(y\zeta_t)^{\frac{1}{\gamma}}} N\left(d_1\left(\underline{\zeta}\right)\right)$.

Let $Y_{T-t} \equiv -r(T-t) - \lambda(w_T - w_t) - \frac{1}{2}\lambda^2 (T-t)$, therefore

$$\zeta_T = \zeta_t e^{Y_{T-t}}$$

where Y_{T-t} has the distribution$Y_{T-t} \sim N\left(\left(-r - \frac{1}{2}\lambda^2\right)(T-t), \lambda^2 (T-t)\right)$.

$\zeta_T \leq \underline{\zeta}$ can be re-written as

$$Y_{T-t} \leq \ln \frac{\underline{\zeta}}{\zeta_t},$$

The first term of (A4) can be derived as follows

$$\begin{aligned}
& \int_{-\infty}^{\ln \frac{\underline{\zeta}}{\zeta_t}} \frac{1}{\zeta_t} \zeta_T (y\zeta_T)^{-\frac{1}{\gamma}} \frac{1}{\sqrt{2\pi\lambda^2 (T-t)}} e^{-\frac{1}{2\lambda^2 (T-t)}\left(Y_T + \left(r + \frac{1}{2}\lambda^2\right)(T-t)\right)^2} dY_{T-t} \\
= & \int_{-\infty}^{\ln \frac{\underline{\zeta}}{\zeta_t}} e^{Y_{T-t}} (y\zeta_T)^{-\frac{1}{\gamma}} \frac{1}{\sqrt{2\pi\lambda^2 (T-t)}} e^{-\frac{1}{2\lambda^2 (T-t)}\left(Y_T + \left(r + \frac{1}{2}\lambda^2\right)(T-t)\right)^2} dY_{T-t} \\
= & \int_{-\infty}^{\frac{\ln \frac{\underline{\zeta}}{\zeta_t} + \left(r - \frac{1}{2}\lambda^2\right)(T-t)}{\lambda\sqrt{T-t}}} (y\zeta_T)^{-\frac{1}{\gamma}} e^{-r(T-t)} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}Z^2} dZ \\
= & \frac{1}{(y\zeta_t)^{\frac{1}{\gamma}}} e^{-r(T-t)} \int_{-\infty}^{d_2\left(\underline{\zeta}\right)} e^{-\frac{1}{\gamma}Y_{T-t}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}Z^2} dZ \\
= & \frac{1}{(y\zeta_t)^{\frac{1}{\gamma}}} e^{-r(T-t)} \int_{-\infty}^{d_2\left(\underline{\zeta}\right)} e^{-\frac{1}{\gamma}\left(\lambda\sqrt{T-t}Z - \left(r - \frac{1}{2}\lambda^2\right)(T-t)\right)} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}Z^2} dZ \\
= & \frac{1}{(y\zeta_t)^{\frac{1}{\gamma}}} e^{-r(T-t) + \frac{1}{\gamma}\left(r - \frac{1}{2}\lambda^2\right)(T-t) + \frac{1}{2}\frac{1}{\gamma^2}\lambda^2 (T-t)} \times \\
& \int_{-\infty}^{d_2\left(\underline{\zeta}\right)} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(Z^2 + 2\frac{1}{\gamma}\lambda\sqrt{T-t}Z + \frac{1}{\gamma^2}\lambda^2 (T-t)\right)} dZ
\end{aligned} \tag{A6}$$

Let $H(T-t) = Z + \frac{1}{\gamma}\lambda\sqrt{T-t}$. (A6) can be re-written as

$$
\begin{aligned}
& \frac{1}{(y\zeta_t)^{\frac{1}{\gamma}}} e^{\Gamma_t} \int_{-\infty}^{d_2(\underline{\zeta})+\frac{1}{\gamma}\lambda\sqrt{T-t}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}H^2} dH \\
= & \frac{1}{(y\zeta_t)^{\frac{1}{\gamma}}} e^{\Gamma_t} N(d_1(\underline{\zeta}))
\end{aligned}
$$

where

$$
Z = \frac{Y_{T-t} + \left(r - \frac{1}{2}\lambda^2\right)(T-t)}{\lambda\sqrt{T-t}},
$$

$$
d_2(x) = \frac{\ln \frac{x}{\zeta_t} + \left(r - \frac{1}{2}\lambda^2\right)(T-t)}{\lambda\sqrt{T-t}}, \ x = \underline{\zeta} \text{ or } \overline{\zeta}
$$

$$
d_1 = d_2(x) + \frac{1}{\gamma}\lambda\sqrt{T-t},
$$

and

$$
\begin{aligned}
\Gamma_t &= -r(T-t) + \frac{1}{\gamma}\left(r - \frac{1}{2}\lambda^2\right)(T-t) + \frac{1}{2}\frac{1}{\gamma^2}\lambda^2(T-t) \\
&= \frac{1-\gamma}{\gamma}\left(r + \frac{\lambda^2}{2}\right)(T-t) + \left(\frac{1-\gamma}{\gamma}\right)^2 \frac{\lambda^2}{2}(T-t).
\end{aligned}
$$

With the same method, we can also prove the remaining two terms of (A4). ■

Chapter 3 Annuitization and Retirement Timing Decisions

I Introduction

In recent years, there has been a significant shift from Defined Benefit (DB) to Defined Contribution (DC) pension plans in a number of countries, including the U.S., the U.K. and Australia. In the U.S., the number of DB plans has declined sharply in recent years, from 112,208 in 1985 to about 29,600 in 2004 (FDIC 2006). In the U.K., DC plans started widely about two decades ago. At 2002, approximately a third of pension schemes in the U.K. are DC and the trend away from DB funds is expected to accelerate in coming years (Ross and Wills 2002). This shift makes it increasingly interesting to understand determinants of DC pension plan participants' retirement decisions.

Retirement decisions of individuals with DC plans are jointly influenced by many factors, for example, expected and realized investment returns, the individuals' risk aversion, the mortality rate, the subjective valuation of leisure, the labor income and its expected growth rate. DC pension plans generally provide benefit in the form of a lump-sum payment. In some countries, for example, the U.S., there are no obligations to annuitize DC wealth, while in others, for example, the U.K., there are obligations to do so. The seminal paper of Yaari (1965) argues that, in the absence of a bequest motive, all retirement wealth of risk-averse individuals should be annuitized[1]. There are two reasons supporting this view. One is that without annuitization there is a risk that the retirees might consume too much so that they will exhaust their retirement resource before they die. The other one is that some retirees might consume too less while they are alive. These individuals could have consumed more to have better life quality. Thus, an important part of annual DC pension income should be annuity income, especially in countries, like the U.K., where there are obligations to annuitize DC wealth.

According to the 2009 Retirement Confidence Survey conducted by the Employee Benefit Research Institute (EBRI), about half of workers in the U.S. still would like to retire no later than age 65 despite the recent financial crisis. In the U.K., the average retirement age for men (women) was about 65 (62) in 2008, increasing from 64 (61) in

[1] The voluntary annuitization participation rate is very low which is referred to as the annuity puzzle. The desire to annuitize might be weakened by, among others, bequest motive, irreversibility of annuitization decision, precautionary savings over unexpected spending boost (for example, medical expenditure). But in a country like the U.K. where the first-pillar State Pension is low, annuitizing the pension wealth accumulated through the employer-sponsored scheme could provide safety net for retirement income.

1984. Since workers' DC wealth shrinks dramatically during the crisis, the fact that they do not want to prolong their working lives raises concern that they will not have sufficient financial wealth to support their retirement lives. However, individuals typically have large freedom to decide when to annuitize their DC wealth after retirement. This freedom allows individuals to benefit from better financial market performance after retirement. Therefore, such a concern might not be necessary and it could be optimal for some individuals to retire even when the financial market performance is sluggish because they could continue investing their DC wealth in the financial market after retirement and annuitize the DC wealth when the market performs better. With the annuitization timing freedom, the decision to retire becomes a decision to optimally exercise a compound real option. Once the individual retires, he gets the option to annuitize his DC pension wealth. The optimal retirement decision depends on the expected outcome of the annuitization option.

This chapter aims to analyze retirement timing decisions of DC pension plan participants, taking into account the optimal annuitization timing decision. To do so, I will first set up a retirement decision model and develop a forward looking retirement likelihood measure from this model. The retirement likelihood measure describes the probability that an individual will retire within the next few years. In the model, the individual obtains utility from leisure, labor income before retirement and pension benefit after retirement. The DC pension benefit is the income from the annuity which is bought at the optimal annuitization timing.

The retirement likelihood measure is then tested with the English Longitudinal Study of Ageing (ELSA) data. The most important reason why I choose U.K. data is that there is an obligation to annuitize pension wealth before age 75[2]. ELSA is a biannual panel survey among those aged 50 and over (and their younger partners) living in private households in England. For all the individuals who are full-time employed in the wave 1 interviews (conducted in 2002-3), the probabilities of retiring by wave 2 interviews (conducted in 2004-5) are evaluated based only on the information available at wave 1 interviews. The model predictions are compared with the actual retirement ratios and the predictions implied by a Probit model where age, gender, education level and DC wealth are explanatory variables used to explain the retirement decisions reported at the wave 2 interviews. The performance of the retirement likelihood measure, in terms of

[2] The desire to annuitize voluntarily is weakened by, among others, the bequest motive, the irreversibility of annuitization decision, and precautionary savings over unexpected spending boost (for example, medical expenditure). But in a country like the U.K. where the first-pillar State Pension is low, annuitizing the pension wealth accumulated through the employer-sponsored scheme provides a safety net for the retirement income.

the correlations with the actual retirement ratios and the roots of Mean Square Errors, are comparable to the performance of the Probit regression. This result gives strong support to the option model setup in this chapter because the prediction from the option model is out of sample while the prediction from the Probit regression is in-sample. This chapter also quantifies the economic benefits of having annuitization timing freedom. The economic benefit is defined as the percentage difference between the certainty equivalent wealth obtained from optimally choosing the annuitization time on the one hand and annuitizing at the retirement time on the other hand. I show that annuitization timing freedom on average leads to a gain of 1.8% for an individual which retires in one year.

This chapter is related to the literature focusing on the determinants of retirement decisions. A first line of research in this area has investigated the retirement incentives of DB pension plan participants. The seminal paper by Stock and Wise (1990) presents an option value model to describe the retirement decisions of DB plan participants. Their model is very close in spirit to the stochastic dynamic programming model of Rust (1987). Stock and Wise (1990) apply their model to data from a large company. They find that their model can explain very well the actual retirement ratios in that company. They argue that pension wealth is a significant determinant of the retirement probability. Samwick (1998) applies the option model to a national-wide dataset. His research confirms and strengthens the results of Stock and Wise (1990). Sundaresan and Zapatero (1997) link the option value to the lifetime marginal productivity schedule which, given their assumption, is increasing at the beginning of the working life and then starts decreasing. They argue that people will retire when the ratio of DB pension benefit and the current wage reaches certain threshold value. This chapter extends the option value model of Stock and Wise (1990) to the DC plan participants' retirement decision.

A second line of research focuses on differences between impacts of DB and DC pension plans on the retirement decision and pension income. Friedberg and Webb (2005) study Health and Retirement Survey data and find that workers with DC plans are retiring significantly later compared with the ones with DB scheme. Samwick and Skinner (2004) investigate whether DC plans, compared to DB plans, are adequate in providing for a comfortable retirement pension. Their results show that 401(k) plans, a popular U.S. variant of DC plans, can be as good as or better than DB plans in providing retirement income.

A third line of research looks at the interactions among wealth, investment strategies and the retirement decisions. Gustman and Steinmeier (2002) and Coronado and Perozek (2003) study the effect of a positive shock in household wealth including private savings and savings through DC accounts on household members' retirement decision making. These two papers investigate the period in the late 1990s when the stock market was booming in the U.S.. Both papers find that the extraordinary high returns in the stock market increase retirement in the United States. Lachance (2003), Choi and Shim (2006), Farhi and Panageas (2007) and Liu and Neis (2002) study the issue of retirement decision and its implication on the investment choice. Choi and Shim (2006) show that the individual consumes less and invests more in risky assets when he has an option to retire than he should in the absence of such an option. Farhi and Penagear (2007) find that investing for early retirement tends to increase savings and reduce an agent's effective relative risk aversion, thus increasing his stock market exposure.

A fourth line of research investigates the interaction between the optimal retirement age and a number of key factors like the availability of annuity products, life expectancy and wages. Sheshinski (2008) provides a comprehensive analysis on these issues.

This chapter is also related to the literature on optimal annuitization timing. The literature on this topic is relatively small but growing. Milevsky and Young (2002) develop a normative model of when the individuals should annuitize their wealth. Their model is built on Merton (1971) and solved by standard continuous-time technology. Milevsky and Young (2007) argue that in the U.S. annuitization prior to age 65-70 is not optimal even in the absence of any bequest motives. Milevsky, Moore and Young (2006) study the interaction between the annuitization timing decision and the optimal portfolio allocation.

The main contribution of this chapter is to incorporate the optimal annuitization timing decision into a normative model explaining the optimal retirement decision making of DC plan participants. There is no doubt that the annuitization timing has large impact on the size of the DC pension benefit. Therefore, rational individuals with DC plans should take this into account while making their retirement decision. Incorporating the optimal annuitization decision making improves the comprehensiveness of a normative model for optimal retirement timing decision. The empirical findings of this chapter suggest that in reality at least some individuals recognize the value of the freedom in choosing the annuitization timing and incorporate it into their retirement decision making.

The organization of the chapter is as follows. Section 2 describes the retirement decision model. Second 3 discusses the empirical investigation of the model prediction. Section 4 concludes.

II The Retirement Decision Model

The aim of this section is to model the optimal retirement decision of an individual participating in a DC plan, taking into account the optimal annuitization timing. This model will also account for the DB and the state pension plans existing next to the DC pension plan. Currently, we are at time 0 and the individual's current age is F, where $50 \leq F < 75$. He is working full time at time 0. He can retire between time 1, 2, 3, ...and time T where time T is the time when this person turns 75 years old. The oldest age the individual could reach is assumed to be $T_{\max}$ and $T_{\max} > T$. His current DC wealth is W_0. The individual does not have to annuitize his retirement wealth immediately after retirement unless he retires at time T. If he retires before time T, he could annuitize his pension wealth between the retirement date, say t, and T.

Assume that the individual retires at time t, where t could be any time between 1 and T and annuitizes at time t_a, which could be either at or between time t and T. His subsequent pension income, $P(t, t_a)$, consists of annuity income, $A(t, t_a)$, after the individual annuitizes his DC wealth, the amount, $Q(t, t_a)$, withdrawn from his DC wealth before annuitization, the income from current and past DB plans, $CDB(t)$ and $PDB(t)$, and the state pension, $SP(t)$, that is,

$$\begin{aligned} P(t, t_a)_j &= \begin{cases} A(t, t_a) + CDB(t)_j + PDB(t)_j + SP(t)_j, & \text{for } T_{\max} \geq j \geq t_a, \\ Q(t, t_a) + CDB(t)_j + PDB(t)_j + SP(t)_j, & \text{for } t_a > j \geq 1, \end{cases} \\ &= DC(t, t_a) + CDB(t)_j + PDB(t)_j + SP(t)_j, \end{aligned} \tag{1}$$

where

$$DC(t, t_a) = \begin{cases} A(t, t_a), & \text{for } T_{\max} \geq j \geq t_a \\ Q(t, t_a), & \text{for } t_a > j \geq 1 \end{cases}.$$

For any given pairs of t and t_a, $Q(t, t_a)$ is constant over time $(t,\ t_a)$ and $A(t, t_a)$ is constant over time $(t_a,\ T)$. The DB and state pension benefits, $CDB(t)_j$, $PDB(t)_j$ and $SP(t)_j$, are indexed to inflation after retirement. The pension benefits, $A(t, t_a)$, $Q(t, t_a)$, $CDB(t)$, $PDB(t)$ and $SP(t)$ will be discussed below in more detail.

The Financial Market

In this section, the asset universe available to the DC pension plan member for investment purposes will be introduced. There are one stock index and one bond available in the financial market. The diffusion processes of the short term interest rate and the stock index are as follows,

$$\begin{aligned} dr_t &= \kappa_r (\overline{r} - r_t) dt + \sigma_r dZ_{1t} && (2) \\ dS_t &= (r_t + \lambda_s \sigma_s) S_t dt + \sigma_s S_t dZ_{2t}, && (3) \end{aligned}$$

where λ_s is the Sharpe Ratio of stock price, σ_r and σ_s are volatilities of short-term interest rate and stock price, and $\overline{r}$ is the long-term average of the short-term interest rate. The Vasicek process (2) is mean reverting. When the short-term interest rate falls below the long-term average, $\overline{r}$, the short-term interest rate tends to increase towards $\overline{r}$ in the future. When the short-term interest rate is above the long-term average, the short-term interest rate tends to fall towards the long-term average in the future. κ_r determines the speed of the such an adjustment process. Z_1 and Z_2 are two standard Brownian Motions supported by a probability space $(\Omega,\ \mathcal{F},\ P)$ over the finite time horizon $(0,\ T)$ with correlation coefficient ρ. All stochastic processes introduced in this chapter are assumed to be measurable with respect to the augmented filtration $\{\mathcal{F}_t : t \in (0,T)\}$.

From the Vasicek model, we can get the price of the zero-coupon bond at time t with time to maturity h

$$B_t^{(h)} = e^{-a(h) - b(h) r_t}, \tag{4}$$

where

$$a(h) = \left(\overline{r} - \frac{\lambda_r \sigma_r}{\kappa} - \frac{\sigma_r^2}{2\kappa^2}\right)(h - b(h)) + \frac{\sigma_r^2}{4\kappa} b(h)^2,$$

$$b(h) = \frac{1}{\kappa}\left(1 - e^{-\kappa h}\right),$$

and λ_r is the interest rate price of risk. The yield of a zero-coupon bond with time to maturity h, $Y(h)$, is

$$Y(h) = \frac{a(h) + b(h) r_t}{h}. \tag{5}$$

By Ito's lemma, the dynamics of any arbitrary bond prices can be described by

$$dB_t = B_t \left[(r_t + \lambda_r \sigma_{B,t}) dt + \sigma_{B,t} dZ_{1t}\right], \tag{6}$$

where $\sigma_{B,t} = \sigma_r D(r,t)$ and $D(r,t) = -(dB_t/dr)/B_t$ is the elasticity of the bond price with respect to the short interest rate. The elasticity is referred to as the duration of the interest rate contingent claim. Following Munk, et al (2004), it is assumed that the bond available for the investor has a constant duration $D > 0$. This can be thought of as reflecting the duration of the aggregate portfolio of bonds outstanding, or a bond index, where bonds that expire are always substituted with new longer term bonds.

The DC Income

As we have seen before, the DC income, $DC(t, t_a)$, consists of the amount the individual withdraw before annuitization, $Q(t, t_a)$, and annuity income after the annuitization, $A(t, t_a)$. The DC income is jointly affected among other factors by the investment returns, the amount of contributions made to the DC plan and the annuity rates.

Let $W(t, t_a)_j$ denote the individual's DC portfolio wealth at time j, $j \in [t, t_a]$, if the individual retires at time t and annuitizes at time t_a. Assume that the total amount of contributions paid by the individual and his employer to the DC plan is C per year. After retirement, the individual will withdraw $Q(t, t_a)$ per year from his DC wealth before annuitization. A fraction α of his DC assets is invested in the stock index and $1-\alpha$ in the bond. As in Samwick and Skinner (2003), the investment portfolio will be rebalanced annually to keep the weight of the stock and bond at α and $1-\alpha$. The optimal annuitization and retirement dates will be described below. For every possible combination of retirement and annuitization dates, that is, $0 \leq t \leq T$ and $t \leq t_a \leq T$, the individual's DC wealth can be described as follows

$$W(t, t_a)_j = \begin{cases} \left(\frac{\alpha \times \left(W(t,t_a)_{j-1}+C\right)}{S_{j-1}}\right) S_j + \left(\frac{(1-\alpha)\times\left(W(t,t_a)_{j-1}+C\right)}{B_{j-1}}\right) B_j, & 1 \leq j \leq t. \\ \left(\frac{\alpha \times \left(W(t,t_a)_{j-1}-Q(t,t_a)\right)}{S_{j-1}}\right) S_j + \left(\frac{(1-\alpha)\times\left(W(t,t_a)_{j-1}-Q(t,t_a)\right)}{B_{j-1}}\right) B_j, & t < j \leq t_a. \end{cases} \tag{7}$$

The upper part of equation (7) describes the wealth process before retirement and the lower part describes the wealth process after retirement. Before the individual retires, the total amount of DC wealth available for investing is the sum of the previous DC wealth and the new contribution. After the individual retires but before the individual annuitizes his DC wealth, the total amount of DC wealth available for investing is the difference between the previous DC wealth and the amount withdrawn by the individual.

If the individual retires at time t and annuitizes his DC wealth at time t_a, the annuity income, $A(t, t_a)$, which he will receive immediately after annuitization until he dies depends, among others, on the term structure and the amount of DC wealth at the annuitization date, t_a. $A(t, t_a)$ is determined as follows,

$$W(t, t_a)_{t_a} = A(t, t_a)\left[1 + \sum_{j=1}^{T_{\max}-t_a}\left(\frac{1}{\left(1+r_{t_a}^{(j)}\right)^j}\prod_{k=1}^{j} M_k\right)\right](1+p). \tag{8}$$

In eq.(8), p is a load factor which is greater than or equal to zero, obtaining a measure of the “money’s worth”of the annuity. If the load factor is zero, then the annuity contract is actuarially fair and the “money’s worth” equals one. Empirical evidence by Mitchell et. al. (1999) illustrates that the load factor varies between 8% and 20% depending on different assumptions about discounting and mortality tables[3]. M_k denote the probability that the individual is alive at time k, conditional on being alive at time $k-1$ and $M_1 \equiv 1$. $r_{t_a}^{(j)}$ is the j-year interest rate at the time of annuitization.

I assume that the amount, $Q(t, t_a)$, the individual withdraws after retirement but before annuitization equals the amount of annuity income he could get if he annuitizes immediately after retirement, that is,

$$Q(t, t_a) = A(t, t).$$

The DB and state pension incomes are introduced in the following part of this section.

The DB and State Pension Income

If the person retires at time t, where t could be any time between 1 and T, his income from current and past DB plans, $CDB(t)$ and $PDB(t)$, are determined by, among others, the accrual rate, years of membership and labor income, that is,

$$\begin{aligned} CDB(t) &= acc\_rate \times n_t \times Y_t && (9) \\ PDB(t) &= acc\_rate \times n_{past} \times Y_{lastyear} \times \exp(\pi(t - t_{lastyear})), && (10) \\ & \quad \text{with } t_{lastyear} < t, \end{aligned}$$

where acc_rate is the accrual rate, n_t is the number of membership years in the current DB scheme at time t, n_{past} is the number of years in the past DB scheme, π is the annual inflation rate, $t_{lastyear}$ is the last year in the past DB plan, $Y_{lastyear}$ is the individual’s annual gross income during his last year in the past scheme and Y_t is the person’s annual gross income at time t. Thus, the DB plan is of a final salary type and the DB income after retirement is indexed to inflation which is required by law in the U.K. (see Blake 2003). This means if the individual retires at time t, his income afterwards is,

$$\begin{aligned} CDB(t)_j &= CDB(t)\exp(\pi(j-t)), \text{ for } j = t...T_{\max}, \\ PDB(t)_j &= PDB(t)\exp(\pi(j-t)), \text{ for } j = t...T_{\max}. \end{aligned}$$

[3] Finkelstein and Poterba (2002) and Cannon and Tonks (2003) study the actuarial fairness of annuity in the U.K.. Both papers find that the annuity is actuarially fair using annuitant mortality table. However, Finkelstein and Poterba (2002) show that the annuity is unfairly priced using the population mortality table. I use the population mortability table here.

The state pension is also indexed to inflation, therefore, we have

$$SP(t)_j = SP(t)\exp(\pi(j-t)), \text{ for } j = t...T_{\max}. \tag{11}$$

The Optimal Retirement and Annuitization Timing

The utility function is closely related to Stock and Wise (1990). At time 0, the individual is full time employed. The individual can retire between time 1 and T. Looking ahead, he will receive his labor income as long as he keeps working. Once he retires he receives pension income and enjoys the leisure until he dies. At time t, $1 \leq t \leq T$, if the individual retires, his utility of retirement, U_t, is the sum of the utility from labor income, pension benefit and leisure, that is,

$$U_t = \sum_{s=1}^{t-1} \exp(\beta(t-s)) \frac{Y_s^{1-\gamma}}{1-\gamma} + \sum_{s=t}^{T_{\max}} \exp(\beta(t-s)) \frac{(KP(t,t_a)_s)^{1-\gamma}}{1-\gamma} \prod_{k=t}^{s} M_k, \tag{12}$$

where β stands for the subjective discount factor and the parameter K takes into account the disutility of work. Y_s stands for labor income which is deterministic and $P(t,t_a)_s$ is the pension income which is explained in (1). $\prod_{k=t}^{s} M_k$ is the cumulative survival probability from time t to s with $M_t = 1$. The first term of (12) is the accumulation of the utility from labor income at time t and the second term is the sum of the discounted utility from pension and leisure at time t. As in Stock and Wise (1990), the parameter K has two specifications. In the first specification, K is a constant. In the second specification, K is a convex function of current age, F, and $K = k_0\left(\frac{F}{k_2}\right)^{k_1}$ where k_0, k_1 and k_2 are constants[4].

For each of the possible retirement stopping times, the DB and state pension income is determined by (9), (10) and (11). But as we have seen before, the DC pension income, $DC(t,t_a)$, depends not only on when the individual retires but also on when DC wealth is annuitized. This makes the retirement option a compounded real option. Once the individual retires, he obtains the right to exercise his annuitization option. But the retirement decision depends on the expected outcome of the annuitization option. Therefore, in order to find a solution to (12), we first have to find the optimal annuitization timing and thus, the optimal DC pension income, $P(t,t_a^*)$, for all the possible retirement times from year 1, 2, 3 to year T. After that, we could attempt to solve for the optimal

[4]The preference for leisure might also be determined by the intensity of work, desire to travel or spending time on hobbies, etc..K could be individual specific. For simplicity, I assume K to be either flat or age dependent.

retirement timing for eq.(12).

The retirement timing decision is an example of optimal stopping problems with fixed horizon. The optimal stopping problem describes the problem of choosing a time to stop a certain action based on sequentially observed random variables in order to maximize the expected payoff or utility. A random variable τ defined on Ω and taking values in the time set is called a stopping time if the event $\{\tau \leq t\}$ belongs to $\mathcal{F}_t$ for all $t \in (1, T)$. In other words, for τ, to be a stopping time, it should be possible to decide whether or not the event $\{\tau \leq t\}$ has occurred based on the knowledge that are known at time t, i.e., the knowledge in the information set $\mathcal{F}_t$. The stopping time for retirement decisions is called retirement stopping time. The retirement problem can be formulated as finding an optimal retirement stopping time, τ_r^*, from all retirement stopping times, τ_r, with values in $(1,\ T)$,that maximizes the expected discounted utility of retirement at time 1, i.e.,

$$\sup_{1\leq\tau_r\leq T} E_1\left[\exp\left(-\beta\left(\tau_r-1\right)\right)\left(\prod_{k=1}^{\tau_r} M_k\right)U_{\tau_r}\right], \tag{13}$$

where $\prod_{k=1}^{\tau_r} M_k$ is the cumulative surviving probability from time 1 to τ_r with $M_1 = 1$.

The annuitization timing decision is also an example of optimal stopping problems with fixed horizon. The stopping time for annuitization decisions is called annuitization stopping time. The annuitization time, τ_a, must be between retirement time and the deadline for annuitization, that is, $\tau_a \in (\tau_r,\ T)$. The optimal annuitization stopping timing, τ_a^*, is the stopping time that maximizes the expected discounted utility of pension income at retirement time τ_r, with $\tau_r \in (1, T)$, that is,

$$\sup_{\tau_r\leq\tau_a\leq T} E_{\tau_r}\left[\exp\left(-\beta\left(\tau_a-\tau_r\right)\right)B\left(\tau_r,\tau_a\right)\right], \tag{14}$$

where $B\left(\tau_r,\tau_a\right)=\sum_{s=\tau_r}^{T_{\max}}\exp\left(\beta\left(\tau_a-s\right)\right)\left(\prod_{k=\tau_r}^{s} M_k\right)\frac{DC(\tau_r,\tau_a)^{1-\gamma}}{1-\gamma}$ and the product, $\exp(-\beta(\tau_a-\tau_r))\times B\left(\tau_r,\tau_a\right)$, is the sum of the discounted utility of pension income at retirement time τ_r.

The Least Square Monte Carlo (LSM) valuation algorithm developed by Longstaff and Schwartz (2001) is adopted to numerically solve the optimal stopping problem. The LSM algorithm follows the dynamic programming principle and provides a pathwise approximation to optimal stopping rules. At time 0, the stock price and interest rate at time 0 are known but future prices and interest rates are unknown. For each of the exercise dates, $1, ...T$, N paths of stock prices and short-term interest rates are simulated. At time t and path i, where t could be any time between 1 and T and i could be any of the simulated paths, the individual would retire or annuitize if the utility of retiring or annuitizing at time t and path i is larger than the expected utility of retiring or annuitizing later conditional on the information available at time t and path i. The LSM algorithm guarantees that there will be one and only one optimal stopping time for each path which solves (13) for the retirement option and (14) for the annuitization option.[5] Details of the numerical solution and the LSM algorithm are provided in Appendix B.

In U.K., the first pillar state pension is very low, which is about GBP80 per week. Annuitizing second-pillar employer-sponsored pension wealth can provide a source of safe retirement income. With respect to private wealth, some individuals may find it attractive to annuitize their private wealth to provide more income and some not due to bequest motives, desires to save for unexpected large expenditures and other reasons. For simplification, both bequest motives and private wealth are not considered in this model.

The Retirement Likelihood Measure

The probability estimated at time 0 of retiring before time k, k could be any time between 1 and T, can be computed as follows. Let $\tau^*_{r_i}$ denote the optimal retirement time for path i, $i = 1, 2, ..., N$. Let H be a $N \times T$ matrix, where the rows correspond to the simulated paths and the columns correspond to time. The matrix H records the optimal retirement decisions of the individual. If $H(i, j) = 1$, j is the optimal retirement time for path i, otherwise, j is not the optimal retirement time for path i, that is,

$$H(i,j) = \begin{cases} 1 & \text{if } j = \tau^*_{r_i} \\ 0 & \text{otherwise} \end{cases} . \tag{15}$$

By construction, there will be only one "1" in each row.

From the optimal decision matrix, H, we can derive an estimator of the probability of retiring before and including time k, $k > 0$. The notation, $PR_0^{OptionModel}$, denotes the retirement probability and

$$PR_0^{OptionModel} = \frac{1}{N} \sum_{j=1}^{k} \sum_{i=1}^{N} H(i,j) . \tag{16}$$

[5] Clément et al. (2002), Egloff (2005) and Moreno and Navas (2003) provide proofs for the convergence of the LSM algorithm.

At time 0, the probability that the individual will retire before and including time k is the percentage of the paths where the optimal retirement times occur no later than time k. This probability is referred to as the retirement likelihood measure.

III Retirement Decisions in the UK

In this section, the economic benefit of annuitization freedom will be evaluated and the likelihood measure developed in the previous section will be tested empirically. The empirical investigation is based on data from the English Longitudinal Study of Ageing (ELSA).

Data

ELSA is a biannual panel survey among those aged 50 and over (and their younger partners) living in private households in England. The field work for ELSA wave 1 is conducted in 2002 -3 and for wave 2 in 2004 -5. There are 12,100 individuals interviewed in wave 1. 1,659 individuals are employed full time (not less than 30 hours per week) and are interviewed again in wave 2. Among them, 518 persons participate in DC plans and provides complete information about their DC accounts. The sample consists of these 518 individuals. Detailed information about the sample selection is given in table 1.

In this sample, 29 persons retired by the wave 2 interviews of ELSA. None of the 29 persons report that their main reason of retirement is due to the sickness of themselves or their family members. 69.5% of the individuals are contracted out which means that they cannot get retirement income from the second pillar state pensions[6]. In addition to the DC schemes, 31.27% of the individuals in the sample also have past DB plans and 11% of the individuals have current DB plans.

Our sample consists of 374 men and 144 women. 18.3% of the individuals have higher education or equivalent degrees. 30.5% of them didn't receive high school education. The summary statistics of the DC plan participants' age, gross income, DC wealth, asset income, benefit income, gross household wealth and debt are presented in table 2. The average age of the sample members is 55. The average annual gross income is about £24,400 and the average DC wealth is £33,122. Overall, the size of the average DC plan is small compared with the gross income. The small size could be caused by the short contribution records and the contributions to parallel pension plans, for example, DB plans and state pensions. DC pension plans started widely in the U.K. in the 1990's, which

[6] A brief introduction to the U.K. pension system is provided in Appendix A.

Table 1: Sample Selection. This table shows detailed information about the sample selection, including the reasons why individuals are removed from the sample and the number of individuals removed. Except reason (5), the other removal reasons are based on information reported at wave 1 interviews. There are 12,100 individuals participated in the ELSA wave 1 interviews. The sample in this chapter consists of 518 individuals with DC pension plans.

Removal Reasons	**Number of Individuals Removed**
(1) Younger than 50 or older than 90	673
(2) Not employed	8375
(3) Work less than 30 hours per week	873
(4) Incomplete information about education	148
(5) Do not participate in wave 2 interviews	372
(6) Do not have DC pension plan	1131
(7) The size of the DC scheme is not known	10

Table 2: Summary Statistics for the DC Plan Participants

	Mean	**Standard Deviation**
Current Age	55.24	3.55
Gross Annual Income (in £)	24,443.95	22,955.85
DC Wealth (in £)	33,121.53	49,371.40
Asset Income (in £)	1,635.14	7,926.75
Benefit Income (in £)	366.16	1,188.48
Household Wealth excl. Primary House (in £)	88,611.31	200,977.47
Debt (in £)	3,278.42	7,639.93

means that the individuals in our sample started to contribute to the DC plan in their 40's. Asset income, benefit income, gross household wealth excluding the primary housing and debt are at household level. Asset income consists of interest income, dividend income and the rent from second house, etc. Benefit income refers to state benefits, for example, Minimum Income Guarantee (MIG), Child Benefit and Disable Benefit. Gross household wealth is the household's overall wealth excluding the house where they live.

I use quarterly data covering the period January 1984 to December 2002 for the variables describing the financial market. For the short rate, I use U.K. Treasury Bill data from Datastream. I obtain the yield of a 10-year U.K. government zero-coupon bond from the Bank of England. For the return on stocks I use the total return (including distributions) on a broad U.K. stock market index constructed by Datastream. The summary statistics for these variables are provided in table 3. The average return on stocks is 15.78%. The average yield on the 10-year zero-coupon bond is 8.45%. The average short rate is only slightly smaller (8.35%) during this particular sample period.

I use the Euler-Maruyama method to discretize the diffusion processes of the short rate, bond price and stock index. The parameters of these diffusion processes are estimated from the U.K. data discussed above. The estimation method is introduced briefly in Appendix C. The estimation results are as follows, $\kappa_r = 0.0232$, $\overline{r} = 0.0129$, $\sigma_r = 0.0019$, $\lambda_s = 0.1639$, $\sigma_s = 0.0923$ and $\lambda_r = -0.0992$. The negative interest rate risk premium is in line with the literature (c.f. Brennan and Xia 2002). The interest rate risk premium is

Table 3: Summary Statistics for the Short-Term Interest Rate, 10 - Year Government Bond Yield and the Stock Market Return in the U.K. 1984 - 2002

	Mean	**Standard Deviation**
3-Month UK T-Bill Rate	8.35%	3.00%
10 - Year Zero - Coupon UK Government Bond Yield	8.45%	2.37%
Annual Return of the U.K. Stock Index	15.78%	18.10%

negative because investors are averse towards increases in interest rates while concerning stocks, investors are averse towards decreases in stock prices (De Jong, Schotman and Werker 2008).

Projected Annual Incomes

Information on past and future gross incomes is necessary to calculate the state pension and the DB pension income. The past and future gross income is projected based on the following variables: a gender dummy, experience which is defined as current age less the age the individual started to work divided by 10, dummies for education degrees and years of schooling. The (log) current gross annual income is regressed on the above mentioned variables and the square term of experience. The sample for testing the retirement likelihood measure consists of 518 individuals who work full time and have DC plans with complete information. But this analysis is based on the 1659 individuals who are working full time as reported at wave 1 interviews in order to make the projection more precise. The regression results are presented in table 4.

The regression results show that female workers earn significantly less than male workers. Individuals with high education degree (higher education degree or equivalent) earn significantly more than individuals with low (lower than high school degree) and medium education (high school degree) degrees. Income also increases with years of schooling. Experience and its square term have correct signs but they are both insignificant which could due to the fact that the individuals in the sample are of similar age.

In this chapter the inflation rate, π, is assumed to be constant at 2% level. The

Table 4: Projection of Labor Income. The past and future gross incomes are projected based on the following variables: a gender dummy, experience which is defined as current age minus the age the individual started to work divided by 10, dummies for education degrees and years of schooling. High Education is a dummy, which equals to 1 if the individual has higher education or equivalent degree. Low Education is a dummy, which equals to 1 if the individual has educational degree lower than high school. The dependent variable is the log of the current gross annual income. Two stars means significance at 5 percent level.

Parameters	**Values**	
Constant	9.5517	**
Female	−0.3806	**
Experience	0.0818	
Experience^2	−0.028	
Low Education	−0.2094	**
High Education	0.3424	**
Years of Schooling	0.0519	**

projected past or future labor incomes for individual i, $Y_{projected}$, is

$$Y_{projected,i} = EY\left(\theta\right)_i \exp\left(\pi\left(\theta - F_i\right)\right),$$

where F_i is the individual i's current age, θ stands for individual i's future age, $\theta > F_i$, or past age, $\theta < F_i$, and $EY(\theta)_i$ denote the projected labor income of individual i at wave 1 interviews if he is θ years old at that time which is derived from the regression reported in table 4.

The State Pension[7]

The amount of state pension the individuals can receive depends on, among others, whether they are contracted out of the second-pillar state pension system and how long they have contributed to the state pension. The individual cannot receive their state pension until his State Pension Age is reached. If the individual delays receiving the state pension, the amount of pension is increased, at present, by approximately 7.5 per cent per year of delay in return. The maximum reward for deferment is 37 per cent, which is achieved by deferring for five years.

For the individuals who contracted out (in) in the wave 1, I assume that they contracted out (in) throughout their working life. Before 2002, the second pillar state pension is called SERPS. After 2002, the SERPS is replaced by S2P. But since S2P is only introduced in 2002, the individuals' contribution records to S2P are very short. Therefore, this reform does not have big impact on the individuals' pension income at 2004. Thus, in this chapter, this reform is ignored. Department of Work and Pensions (2005) gives a very detailed description about the calculation of the first pillar state pension income (BSP) and the second pillar state pension income (SERPS) which is adopted for the calculation of state pension in this chapter.

The Economic Benefits of Annuitization Freedom

I use the certainty equivalent gain to quantify the economic benefit of annuitization. The certainty equivalent gain is defined as the percentage difference between the certainty equivalent wealth at time 0 of retiring in the future with and without annuitization freedom. For the case without annuitization freedom, the individual has to annuitize at the retirement date. In the sample, the average discounted certainty equivalent gains are about 1.8%, 1.2% and 1.0% for individuals who retire at year 1, year 2 and year

[7] Please see Appendix A for an introduction to the U.K. pension system.

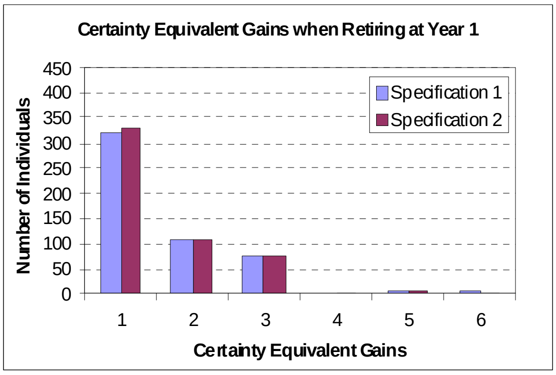
3, respectively. The distributions of the certainty equivalent gains are shown in figure 1. The certainty equivalent gain distributions are almost identical for the two leisure parameter specifications. Assuming that all 518 individuals in the sample retire at year 1, certainty equivalent gains between 0% and 2% (the first column) are experienced by 320 individuals. There are about 100 individuals whose certainty equivalent gains are between 2% and 4% (the second column). The later the retirement date the smaller the certainty equivalent gains due to time preference, risk aversion and the shortening of the annuitization option life.

The Estimated Retirement Probability

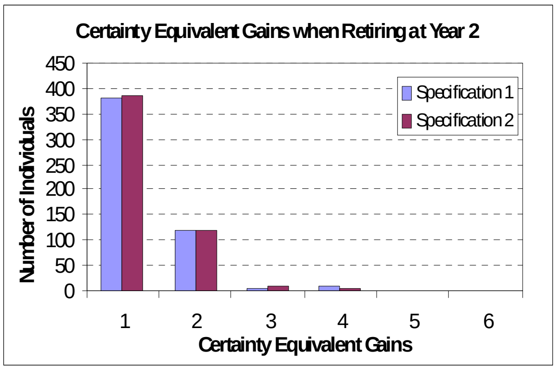
It is assumed that the interviews of wave 1 are conducted at the end of 2002 and the interviews of wave 2 are conducted at the end of 2004. For the individuals who are reported to be retired at wave 2 interviews, the exact retirement years are not known. Based on the information available at wave 1 interviews, the probabilities of retiring by wave 2 interviews, $PR_{2002}^{OptionModel}$, are estimated for every individuals in the sample from eq.(16).

I assume that during 2003 and 2004, at the beginning of each year the individual has a chance to consider retirement. The stock prices and bond prices for the years 2003 and 2004 are simulated from the diffusion processes (3) and (6). The value of the parameters of these diffusion processes are estimated from the market prices before and up to the end of 2002. 2000 paths for future stock and bond prices are simulated. The subjective discount factor, β, is set to 0.03, the risk aversion parameter, γ, is 5, and the preference for leisure, K, equals to 1.5 in specification 1. In specification 2, the preference for leisure K equals to $\left(\frac{F}{55}\right)^5$. 70% of the portfolio assets are stocks and 30% are bonds, that is, $\alpha = 0.7$. The load factor in the annuity price calculation (8) is assumed to be 0.2. The mortality rates are obtained from the U.K. Government Actuary's Department (GAD). The maximum age individuals can live is assumed to be 100.

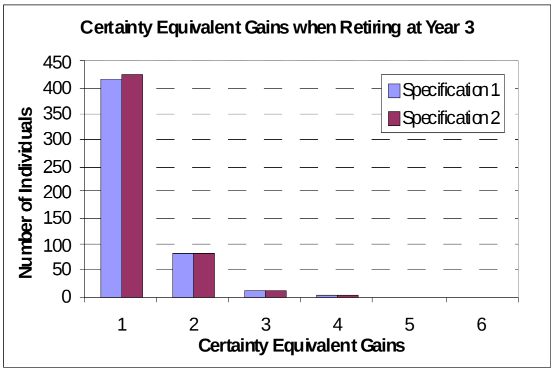
Table 5 reports actual and average predicted percentages of individuals who retire during 2003 and 2004 for the whole sample (518 individuals) and two subsamples. Subsample 1 consists of individuals who were retired by wave 2 and subsample 2 consists of individuals who were not yet retired by wave 2. The actual percentage of retirement is 5.6%. The predicted percentage of retirement is 7.75% for specification 1 and 6.14% for specification 2. The predicted percentage of retirement for subsample 1 is 20.57% for specification 1 and 33.10% for specification 2. For subsample 2, the predicted percentage of retirement is 6.99% for specification 1 and 4.54% for specification 2.



Panel a: The certainty equivalent gains when retiring at year 1



Panel b: The certainty equivalent gains when retiring at year 2



Panel c: The certainty equivalent gains when retiring at year 3

Figure 1: The Certainty Equivalent Gain of Annuitization Freedom. This figure shows the certainty equivalent gain of annuitization freedom. The certainty equivalent gain is defined as the percentage difference between the certainty equivalent wealth at time 0 of retiring in the future with and without annuitization freedom. For the case without annuitization freedom, the individual has to annuitize at the retirement date. The certainty equivalent gain scale 1 refers to the interval [0%, 2%], scale 2 refers to [2%, 4%], scale 3 refers to [4%, 6%], scale 4 refers to [6%, 8%], scale 5 refers to [8%, 10%], and scale 6 refers to [10%, $+\infty$]. In specification 1, the leisure parameter is a constant, $K = 1.5$, and in specification 2 the leisure parameter is age dependent, $K = (F/55)^5$. Panel a, b and c show the certainty equivalent gains under the assumption that all 518 individuals retire at year 1, year 2 and year 3, respectively.

Table 5: The predicted retirement probability evaluated at the end of 2002 is an indicator measuring how likely the individuals will retire between 2003 and 2004. It is evaluated with the method discussed in section 2. This table reports the mean of the estimated retirement probability for the whole sample, the subsample consisting of individuals retired at wave 2, and the subsample consisting of individuals who are not retired at wave 2. There are two specifications for parameter K. In model specification 1, the disutility of work parameter K is a constant which equals to 1.5. In model specification 2, the disutility of work parameter is age dependent.

	Actual Percentage of Retirement	Predicted Retirement Probability $PR_{2002}^{OptionModel}$	
		$K = 1.5$	$K = (F/55)^5$
Whole Sample	5.60%	7.75%	6.14%
Subsample 1: Retired	100.00%	20.57%	33.10%
Subsample 2: Not Retired	0.00%	6.99%	4.54%

The Proxy of Retirement Incentive

In order to check whether the retirement likelihood measure, $PR_{2002}^{OptionModel}$, is significant in explaining and predicting the retirement decision making in reality, the retirement likelihood measure is treated as a proxy for retirement incentives. A Probit analysis is applied to test the significance of this proxy[8]. The dependent variable is the sample individuals' retirement decisions reported at wave 2 interviews which takes value 1 if the individual is reported to be retired and 0 if not. The variables, Asset Income (AI), Benefit Income (BI), Gross Household Wealth (GH) and Debt, which are not used for calculating $PR_{2002}^{OptionModel}$ are also included in the analysis. The results are presented in table 6. For both leisure parameter specifications, the proxy of retirement incentives, $PR_{2002}^{OptionModel}$, is positive and significant at 1% level no matter whether the other four variables are included or not. This analysis shows that the retirement likelihood has significant explanatory power in explaining and predicting the retirement decision in reality. It also means that financial incentives are important to the DC plan participants when they are making their retirement decision.

The Model Fit

The model fit is analyzed by comparing the actual retirement probability at wave 2 interviews, the predicted retirement probability from the option model based on wave 1 interview information, $PR_{2002}^{OptionModel}$, and the predicted retirement probability from a Probit model, $PR_{2002}^{\Pr obit}$, where the regressors are variables such as, age, gender, education dummies, gross income and DC wealth, which are used for evaluating the retirement likelihood measure $PR_{2002}^{OptionModel}$ and the dependent variable is the retirement decision at wave 2. The probability of retiring by wave 2 interviews computed from this Probit model is actually an in-sample prediction. By contrast, the prediction from the option model is out of sample.

The Probit regression reported in table 7 shows the impact of these variables on the individuals' retirement decision in the sample. The results are very intuitive. Older individuals are significantly more likely to retire than younger ones. Women are significantly more likely to retire than men. This is because in the U.K., the State Pension Age for women at 2002 is lower than that for men. Age and gender are significant at 5% level. DC wealth, gross income and education dummies have expected signs, but they are insignificant. From the Probit model in table 7, for each individual we can compute the

[8] The estimation error of the estimated retirement likelihood measure, $PR_{2002}^{OptionModel}$, is not taken into account when estimating the standard deviation of the estimated slope coefficient of this variable.

Table 6: Retirement Likelihood Measure As a Proxy for Retirement Incentive. This table reports the results of the Probit regression where the retirement probability derived from the option model is cheated as a retirement incentive. The dependent variable equals to 1 when the individual is reported to be retired at wave 2 and 0 otherwise. Asset income (AI) consists of interest income, dividend income and the rent from second house, etc. Benefit income (BI) refers to the state benefits, for example, Minimum Income Guarantee (MIG), Child Benefit and Disable Benefit. Gross household wealth (GH) is the household's overall wealth excluding the house where they live. There are two specifications for parameter K. In model specification 1, the disutility of work parameter K is a constant which equals to 1.5. In model specification 2, the disutility of work parameter is age dependent. Panels A and B report the results from specification 1 and 2, respectively. One star stands for significance at 10 percent level and two stars stand for significance at 5 percent level and three stars stand for significance at 1 percent level.

	Submodel 1		**Submodel 2**	
Panel A: Specification 1				
Constant	-1.6753	***	-1.6612	***
	(0.0991)		(0.1193)	
$PR_{2002}^{OptionModel}$	0.7288	***	0.7554	***
	(0.2820)		(0.2895)	
Asset Income (AI) / £1000			-0.0075	
			(0.0177)	
Benefit Income (BI) / £100			0.0045	
			(0.0064)	
Gross Household Wealth (GH) / £10000			0.0024	
			(0.0061)	
Debt/£1000			-0.0168	
			(0.0181)	
Panel B: Specification 2				
Constant	-1.7709	***	-1.7586	***
	(0.1049)		(0.1249)	
$PR_{2002}^{OptionModel}$	1.3700	***	1.4211	***
	(0.2687)		(0.2794)	
Asset Income (AI) / £1000			-0.0115	
			(0.0210)	
Benefit Income (BI) / £100			0.0061	
			(0.0064)	
Gross Household Wealth (GH) / £10000			0.0010	
			(0.0068)	
Debt/£1000			-0.0013	
			(0.0017)	

Table 7: Comparison Probit Regression. This table reports the results of the Probit regression of age, gender and other variables related to individual retirement decisions. The dependent variable equals to 1 when the individual is reported to be retired at wave 2 and 0 otherwise. The gender dummy equals to 1 for woman and 0 for man. The high education dummy equals to 1 for the individuals with higher education degree or equivalent. The low education dummy equals to 1 for the individuals with degree lower than high school degree. One star stands for significance at 10 percent level and two stars stand for significance at 5 percent level and three stars stand for significance at 1 percent level.

	Coefficients	
Constant	-11.0969	***
	(1.6676)	
Age	0.1642	**
	(0.0281)	
Gender	0.5301	**
	(0.2331)	
High Education	-0.5086	
	(0.3671)	
Low Education	0.0728	
	(0.2242)	
DC Wealth/1000	0.0027	
	(0.0021)	
Gross Income/1000	-0.0025	
	(0.0067)	

(in-sample) probability of retiring by wave 2 interviews, $PR_{2002}^{\Pr obit}$.

As in Stock and Wise (1990), I divide the sample into several age groups and then compare the actual retirement ratio in each age group with the predictions from the option model, $PR_{2002}^{OptionModel}$, and from the Probit model, $PR_{2002}^{\Pr obit}$. The results are shown in figure 2 and table 8. It can be seen from figure 2, that the actual retirement probability increases with age. The predictions from the option model catches this trend very well especially the one from specification 2. The correlations between the option model probabilities and the actual retirement probabilities are 0.89 for model specification 1 ($K = 1.5$) and 0.94 for model specification 2 ($K = \left(\frac{F}{55}\right)^5$). The correlation between the (in-sample) Probit model probabilities and the actual probabilities is 0.94. Furthermore, the option model probabilities have roughly the same roots of Mean Square Errors (MSEs) as those from the Probit analysis. The root of the MSE is 6% for the Probit model, 7% for the option model specification 1 and 5% for the option model specification 2.

The sample was also divided by the DC wealth level. Level 1 includes the individuals with DC wealth smaller than $\pounds 5,000$. Level 2 includes individuals with DC wealth larger than $\pounds 5,000$ but smaller than $\pounds 10,000$ and so on until level 7 which is the highest level and includes the individuals with DC wealth larger than or equal to $\pounds 150,000$. The results are reported in figure 3 and table 9. Overall, the actual retirement probability is increasing with the DC wealth level. The correlation coefficient between the actual retirement ratio (column 2 in table 9) and the predicted retirement ratio from the Probit model is 0.78. The correlation coefficient between the actual retirement ratio and the predicted retirement ratios from the option models are about 0.72 for specification 1 and 0.67 for specification 2. The root of the MSE of the Probit model is 2%. The roots of the MSEs of option model specifications 1 and 2 are 27% and 19%, respectively. The relatively large MSEs are due to the prediction errors for the very wealthy groups (group 6 and 7). There are 39 individuals in these two groups. The roots of the MSEs of option model specifications 1 and 2 without these individuals are about 3%.

The prediction errors might be explained by other factors which are not captured by this model, e.g., partner's retirement decision, riskness of labor income and labour intensity of the job. It is well documented that couples prefer to retirement together or very shortly one after the other (c.f., Gustman and Steinmeier, 2000, 2002 and 2004). An unexpected salary cut can also increase the incentive to retire. Furthermore, individuals with more labor intensive work might want to retire earlier than others.

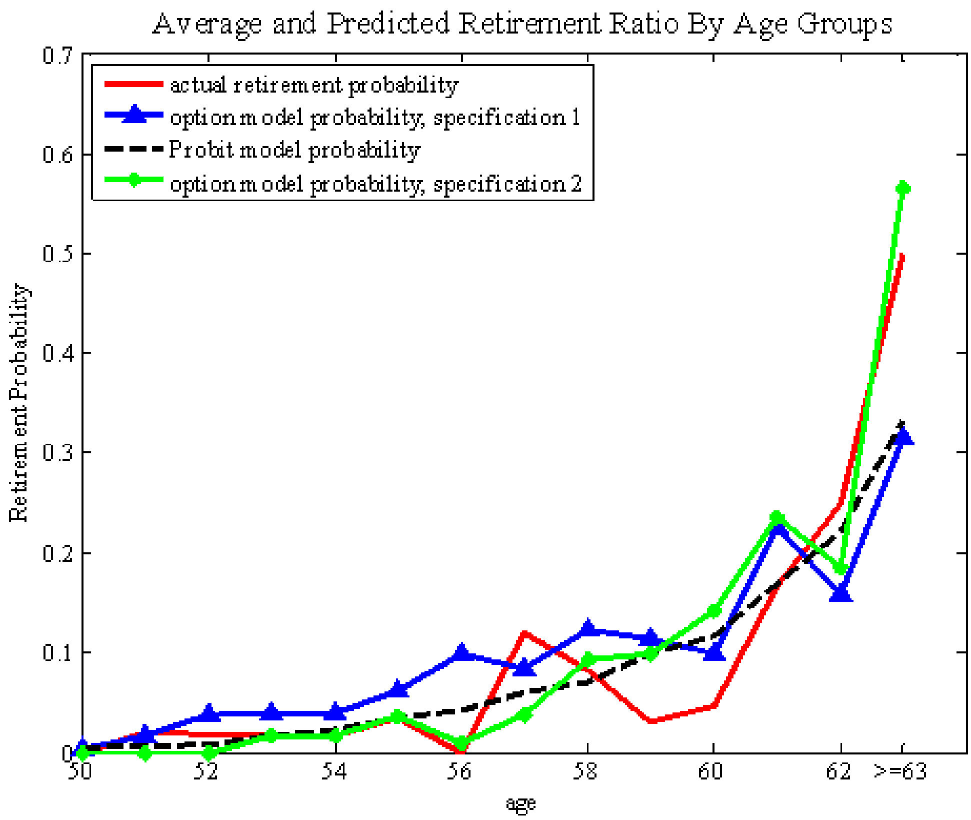


Figure 2: Actual and Predicted Retirement Ratios By Age Groups. This figure shows the actual and predicted retirement ratios from the Probit model and the option model which are reported in table 8.

Table 8: Actual and Predicted Retirement Ratios By Age Groups. This table shows the actual and predicted retirement ratios from the Probit model presented in table 7 and the option model described in section 2. The actual percentage of retirement measures the percentage of individuals retired by the end of 2004 for each age group. In the option model, there are two specifications for the leisure parameter, K. In specification 1, the disutility of work parameter K is a constant which equals to 1.5. In model specification 2, the disutility of work parameter is age dependent.

Age	No. of Obs	Actual Percentage of Retirement	*In-Sample* Probit Model Prediction	*Out-of-Sample* Option Model Prediction	
				$K = 1.5$	$K = (F/55)^5$
50	29	0.00	0.00	0.00	0.00
51	50	0.02	0.01	0.02	0.00
52	53	0.02	0.01	0.04	0.00
53	56	0.02	0.02	0.04	0.02
54	59	0.02	0.02	0.04	0.02
55	58	0.03	0.03	0.06	0.04
56	50	0.00	0.04	0.10	0.01
57	25	0.12	0.06	0.08	0.04
58	36	0.08	0.07	0.12	0.09
59	33	0.03	0.10	0.11	0.10
60	21	0.05	0.12	0.10	0.14
61	18	0.17	0.17	0.23	0.24
62	12	0.25	0.22	0.16	0.19
>= 63	18	0.50	0.33	0.31	0.56
Corr. Coef. With Column 3			0.94	0.89	0.94
Root of Mean Squared Error			0.06	0.07	0.05

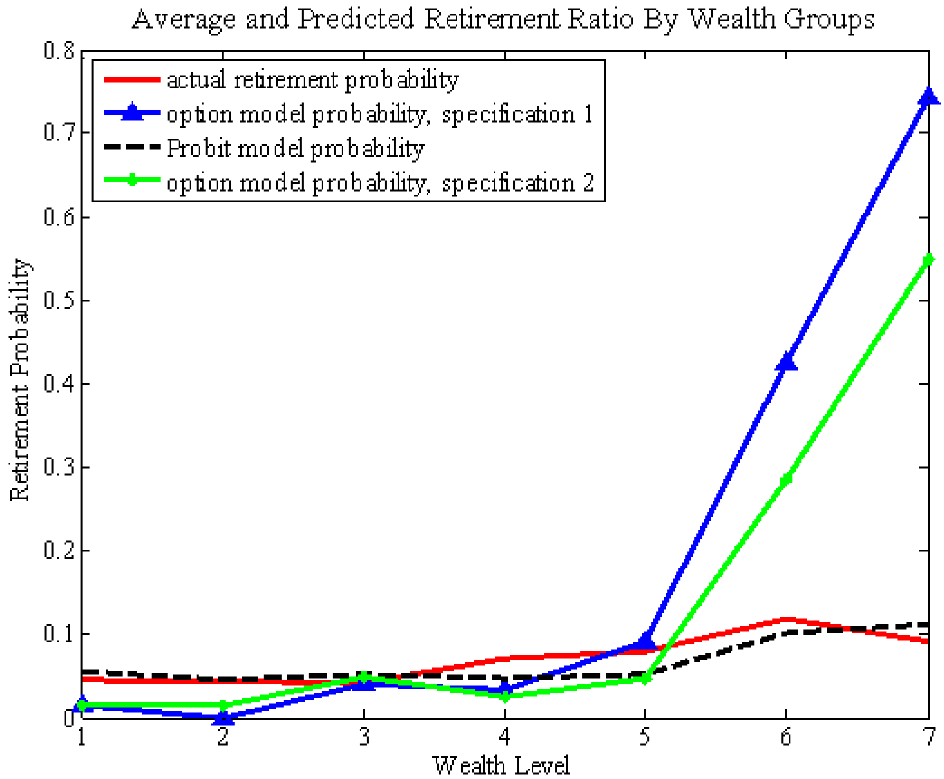


Figure 3: The Actual and Predicted Retirement Ratios By DC Wealth Group. This figure shows the actual and predicted retirement ratios from the Probit model and the option model. The actual percentage of retirement measures the percentage of individuals retired by the end of 2004 for each wealth group. The Probit model is described in table 7. The option model is described in section 2. Group 1 are the individuals with DC wealth less than £5,000, Group 2 includes the individuals with DC wealth between £5,000 and £10,000, Group 3 includes the individuals with DC wealth between £10,000 and £25,000, Group 4 includes the individuals with DC wealth between £25,000 and £50,000, Group 5 includes the individuals with DC wealth between £50,000 and £100,000, Group 6 includes the individuals with DC wealth between £100,000 and £150,000, and Group 7 includes the individuals with DC wealth larger or equal to £150,000.

Table 9: Actual and Predicted Retirement Ratio By DC Wealth Groups. This table shows the actual and predicted retirement ratios from the Probit model and the option model. The actual percentage of retirement measures the percentage of individuals retired by the end of 2004 for each wealth group. Probit model is described in table 7 and the prediction from the option model is described in section 2.

Level	DC Wealth (in £1000)	No. of Obs	Actual Ret. Percentage	*In-Sample* Probit Model Prediction	*Out-of-Sample* Option Model Prediction	
					$K = 1.5$	$K = (F/55)^5$
1	< 5	132	0.05	0.06	0.02	0.01
2	$[5, 10]$	70	0.04	0.05	0.00	0.01
3	$[10, 25]$	145	0.04	0.05	0.04	0.05
4	$[25, 50]$	56	0.07	0.05	0.03	0.02
5	$[50, 100]$	76	0.08	0.05	0.09	0.05
6	$[100, 150]$	17	0.12	0.10	0.42	0.28
7	$>= 150$	22	0.09	0.11	0.74	0.55
Corr. Coef with Column 4				0.78	0.72	0.67
Root of Mean Square Error				0.02	0.27	0.19

Generally speaking, the performance of the option model, especially using the model specification where the leisure parameter is age dependent, in terms of correlations with the actual retirement probabilities and the roots of the Mean Square Errors, are comparable to the performance of the in-sample Probit predictions. However, for the groups with large pension wealth, the retirement ratios per group given by the model are much higher than the actual retirement ratios. In this model, the desire for leisure is increasing with retirement income. That relationship might need to be adjusted after certain wealth level.

DC As A Main Retirement Income Source

This subsection studies individuals whose main retirement income is from DC. I select the individuals by comparing (1) the sum of the present value of non-DC income taking into account the mortality rate if they retire immediately after wave 1 interviews with (2) the DC wealth reported at wave 1 interviews. Individuals enter this subsample if the DC wealth is larger than the sum of the present value of non-DC income. There are 108 individuals in this subsample. The leisure parameter in this subsection is age-dependent.

First, the financial condition in which individuals find optimal to retire is also investigated briefly. The lower the interest rate, the higher the annuity income a retired person will get for any given value of pension wealth. The option model where retirement and annuitization occur at the same time predicts that for 27.8% of individuals there are more than 50% of chances that these persons will retire when the short rate is above its long-run average. The option model with annuitization optimization predicts that for 31.5% of individuals there are more than 50% of chances that they will retire when the short rate is relatively high.

Second, the result of the Probit model estimation where the retirement rate predicted by the option model with annuitization optimization is treated as a retirement incentive proxy is presented in table (10). Table (10) also shows the Probit regressions of other retirement incentive proxies, namely, (1) the proxy derived from the model where individuals retire and annuitize at the same time and (2) the proxy derived from the model using scenarios which are most close to realized scenarios in 2003 and 2004. I use the K-nearest neighbour (KNN) method to select 10 scenarios. The marginal effect is used as an indicator for prediction precision. Table (10) shows that all retirement incentives are significant and positive in predicting the actual retirement decision. The predictions from both models with annuitization optimization generate higher marginal effect than the prediction from the model without annuitzation optimization.

IV Conclusions

This chapter models the real options to retire and - conditional on retirement - to annuitize the accumulated wealth in Defined Contribution (DC) pension schemes. It contributes to the extensive literature on the option value of retirement in the tradition of Stock and Wise (1990) but shifts the focus from participants in Defined Benefit (DB) schemes to participants in DC schemes. This accounts for the observation that DB schemes are increasingly being replaced by DC schemes in most industrialized countries including the U.S. and the U.K.. A major contribution of this chapter is to recognize and model the sequence of retirement and annuitization options. Since annuitization does not need to occur at the retirement date in many DC schemes, participants in DC schemes can time the financial markets in order to annuitize in an environment of high prices on the stock market which increase the DC wealth and high interest rates which reduce the price of an annuity. In a model where individuals obtain utility from leisure, labor income before retirement and pension income after retirement, I show that the freedom to optimally choose the annuitization time can lead to an increase of certainty equivalent wealth of up to 1.8%. Hence, the embedded annuitization option in the retirement option value is of significant economic value to individuals.

In order to assess the predictive power of my model, I compare retirement likelihoods derived from the theoretical setup with retirement decisions observed at the second wave of the English Longitudinal Study of Ageing (ELSA) for a sample of individuals who were full-time employed at the time of the first wave interviews. It turns out that the theory-motivated retirement likelihood measure is a statistically significant predictor of actual retirement decisions. Moreover, I show that the proposed retirement likelihood measure is highly correlated with observed retirement ratios across groups of individuals defined by age or wealth. The correlation reaches 94% and is not dominated by the predictions of a retirement Probit model which in contrast to my proposed retirement likelihood measure is based on in sample information. With a magnitude of 5%, root mean square errors turn out to be small. These empirical results suggest that individuals do take into account the embedded annuitization option when they decide on when to retire.

Table 10: Retirement Likelihood Measure As a Proxy for Retirement Incentive. This table reports the results of the Probit regression where the retirement probability derived from three option models, (1) option model with annuitization optimization, (2) option model without annuitization optimization and (3) option model with scenarios selected by K Nearest Neighbour method. The dependent variable equals to 1 when the individual is reported to be retired at wave 2 and 0 otherwise. The leisure parameter is age dependent.

Option Model Prediction With Annuitization Optimization		
Constant	-1.6105	***
	(0.2136)	
$PR_{2002}^{OptionModel}$	1.2664	***
	(0.4335)	
Marginal Effect	0.1449	
Option Model Prediction Without Annuitization Optimization		
Constant	-1.6647	***
	(0.2201)	
$PR_{2002}^{NoOptimization}$	1.4571	***
	(0.4315)	
Marginal Effect	0.1376	
KNN Option Model Prediction With Annuitization Optimization		
Constant	-1.6759	***
	(0.2254)	
PR_{2002}^{KNN}	1.3550	***
	(0.4223)	
Marginal Effect	0.1406	

Appendix A

The British Pension System

The U.K. pension system consists of three main pillars. The first pillar, known as Basic State Pension (BSP), is a mandatory, flat rate state pension[9]. The second pillar system is provided by the state, employers and private sector financial institutions. In the second pillar, the employees have considerable choices over the type of pension that they can accumulate. The main choices are between: (1) an earnings-related state pension plan[10]; (2) an occupational DB plan provided by employers and (3) an occupational DC pension plan. The state pension plan offers a pension that is low relative to average earnings, but is fully indexed to prices after retirement. The occupational DB plan offers a relatively high level of pension to the employees who spend most of their working time with the same employer, but provides poor transfer values between plans on changing jobs. The occupational DC pension plan is fully portable, but the pension income depends on uncertain investment returns (see Blake 2003). The second pillar state pension is by default compulsory to all the employees who earn above a lower threshold set by the state. But individuals are able to contract out of the second pillar state pension into an occupational pension scheme provided that the latter is at least as generous as the second-pillar state pension. The third pillar consists of voluntary private pension plans[11]. The third pillar pension arrangements are usually of DC type.

In the U.K., the DC plan participants do not have to annuitize their DC wealth immediately at the retirement date. Up to one-quarter of the value of a pension fund can be taken as a lump sum, but three-quarters must be annuitized before the age of 75 (Finance Act 1995)[12]. The obligation to annuitize DC wealth and the freedom in choosing the annuitization time are the most important reasons why U.K. data is selected for the empirical investigation in this chapter.

[9] The BSP is funded on a pay-as-you-go basis. It is a flat rate benefit. Individuals are entitled to at least some part of the BSP if they have made National Insurance (NI) contributions for at least 25% of their working lives. The BSP benefit in 2006/7 is about £85 per week (Department of Work and Pensions). This benefit is indexed to inflation (Clark and Emmerson 2003).

[10] The second pillar state pension plan was called State Earnings-Related Pension Scheme (SERPS) and replaced by State Second Pension (S2P) in 2002. The second pillar state pension plans are of DB nature (Cocco and Lopes 2004). Both the first and second pillar state pensions are paid by the Department of Pension and Working once the retiree reaches his State Pension Age (SPA). Currently the State Pension Age is 65 for men and 60 for women. By 2020, the SPA for woman will increase gradually to 65.

[11] Employers and individuals can also make additional contributions to a private pension. The state supports the savings in private pension plans through tax relief (see Clark and Emmerson 2003).

[12] Since April 6, 2006, which is after the second wave interviews of ELSA, the individuals at age 75 can also choose to drawdown their DC wealth without annuitization, known as Alternatively Secured Pension (ASP). But tax charges introduced by the government on ASP make this option very unattractive to retirees.

Appendix B

The Solution Method

The optimal annuitization and retirement decisions are very similar to the decision of exercising an American option optimally, in the sense that, like the American option, both the retirement and annuitization decisions can be made at any stopping time between the "purchase" date, in our cases, the time when the individual is allowed to retire/annuitize, and the "expiration" date, in our cases, the time when the individual turns 75 years old.

Let n be the "purchase" date of an annuitization option or a retirement option. The optimal annuitization and retirement stopping problems can be stated as

$$V_n = \sup_{n \leq \tau \leq T} E_n \left[\exp\left(-\beta\left(\tau - n\right)\right) Z_\tau \right], \tag{17}$$

where the function $Z(\cdot) = \left(\prod_{k=1}^{\tau} M_k \right) U(\cdot)$ for the retirement option, $Z(\cdot) = B(\cdot)$ for the annuitization option, $n = 1$ for the retirement option and $n = \tau_r$ for the annuitization option.

The standard solution to a optimal stopping problem with finite horizon is to follow the dynamic programming principle (c.f. Peskir and Shiryaev 2006). Let J_t be the highest attainable expected utility at time t the individual can achieve if he exercises his option at or later than time t, that is,

$$J_t = \sup_{t \leq \tau \leq T} \exp\left(-\beta\left(\tau - t\right)\right) E\left(Z_\tau | \mathcal{F}_t\right).$$

Here exercising an option means retiring for the retirement option and annuitizing for the annuitization option. At time $t = T$, the individual has to stop immediately and gains $J_T = Z_T$. At time $t = T - \Delta t$, where Δt stands for very short period of time, he can either stop or continue. If he stops, $\tau = t$ and $J_{T-\Delta t}$ equals to $Z_{T-\Delta t}$, and if he continues, $\tau = T$ and $J_{T-\Delta t}$ equals to $\exp\left(-\beta\Delta t\right) E\left(J_T | \mathcal{F}_{T-\Delta t}\right)$. It follows that if $Z_{T-\Delta t} \geq \exp\left(-\beta\Delta t\right) E\left(J_T | \mathcal{F}_{T-\Delta t}\right)$ then he needs to stop at time $t = T - \Delta t$; otherwise, he needs to continue at time $t = T - \Delta t$. This decision rule reflects the fact that the individual's decision about stopping or continuation at time $t = T - \Delta t$ must be based on the information contained in $\mathcal{F}_{T-\Delta t}$ only. For $t = T - 2\Delta t, ..., n$, the considerations are continued analogously.

The method of backward induction just explained leads to a sequence of random variables, $(J_t)_{n\leq t\leq T}$, defined recursively as follows:

$$\begin{aligned} J_t &= Z_T \text{ for } t=T; \\ J_t &= \max\left(Z_t,\ \exp\left(-\beta\Delta t\right)E\left(J_{t+\Delta t}|\mathcal{F}_t\right)\right) \text{ for } t=T-\Delta t,...n. \end{aligned}$$

The method also suggests that we consider the following stopping time

$$\tau_n = \min\left\{n\leq k\leq T : J_k = Z_k\right\} \tag{18}$$

as a candidate for optimal stopping time for problem (17). Peskir and Shiryaev (2006) proved that τ_n is indeed the optimal stopping time in (17). The proof is provided in Appendix C.

At time t, $t<T$, the value of immediate exercise, Z_t, is known to the individual. But the value of $\exp\left(-\beta\Delta t\right)E\left(J_{t+\Delta t}|\mathcal{F}_t\right)$ is still unknown. The key to solve the optimal stopping problem (17) is therefore, to evaluate the conditional expectations, $\exp\left(-\beta\Delta t\right)E\left(J_{t+\Delta t}|\mathcal{F}_t\right)$ for $t=T-\Delta t,...n$. Least Square Monte Carlo (LSM) valuation algorithm developed by Longstaff and Schwartz (2001) is adopted to approximate $E\left(J_{t+\Delta t}|\mathcal{F}_t\right)$ and to solve optimal stopping problem numerically. Clément, Lamberton and Protter (2002), Egloff (2005) and Moreno and Navas (2003) proved the convergence of the LSM algorithm.

The Least Square Monte Carlo (LSM) Algorithm

The objective of the LSM algorithm is to provide a pathwise approximation to the optimal stopping rules. It is assumed that the option can only be exercised and considered at a finite number of discrete times, $n,...,t,\ t+\Delta t,...T$. For each exercise date, $n,...T$, N paths (scenarios) of stock prices and short-term interest rates are simulated.

The LSM algorithm follows the standard backward induction method as described previously. At the final expiration date, T, the option has to be exercised, the individual gets $Z_{T,i}$, where i stands for a simulated path and $i=1,2,...N$. At exercise dates before the final expiration date, say time t, the individual must choose whether to exercise the option immediately or to keep the option alive and make the exercise decision at the next exercise date. At time t, for any path i, where the utility from immediate exercise, $Z_{t,i}$, is larger than or equal to the expected utility of continuation conditional on the information available at time t and path i, $\exp\left(-\beta\Delta t\right)E\left(J_{t+\Delta t}|\mathcal{F}_{t,i}\right)$, it is optimal to exercise the option. For any paths where the opposite holds, it is optimal to wait.

At time t and path i, the value of immediate exercise, $Z_{t,i}$, is known to the individual but the value of waiting, $E\left(J_{t+\Delta t}|\mathcal{F}_{t,i}\right)$, is unknown. The conditional expectation at time t and path i, $\left(-\beta\Delta t\right)E\left(J_{t+\Delta t}|\mathcal{F}_{t,i}\right)$, is approximated by regressing the vector of discounted value of continuation at time t, $\exp\left(-\beta\Delta t\right)\mathbf{J}_{t+\Delta t}$, where $\mathbf{J}_{t+\Delta t} = \left(J_{t+\Delta t,1}, J_{t+\Delta t,2}, ...J_{t+\Delta t,N}\right)'$, on the simulated paths of relevant state variables at time t, $\mathbf{X}_t's$ where $\mathbf{X}_t's$ include the utility at time t and the DC wealth at time t.

Let $\hat{E}^m\left(J_{t+\Delta t}|\mathbf{X}_{t,i}\right)$ denote the estimated conditional expectation at time t and path i. The individual will decide at time t whether to exercise the option or not. For the paths where the value of immediate exercise, $Z_{t,i}$, is larger (smaller) than or equal to the estimated conditional expectation, $\exp\left(-\beta\Delta t\right)\hat{E}^m\left(J_{t+\Delta t}|\mathbf{X}_{t,i}\right)$, it is optimal to exercise the option (wait). Proceed these calculations and comparisons recursively backwards until the "purchase" date is reached. The optimal stopping time for each path is then decided by starting from the "purchase" date, moving along each path until the first stopping time. For each path, the first stopping time is the optimal exercise time for that path. Thus, there will be one and only one optimal stopping time for each path.

Appendix C

We have

$$J_n \geq E\left(Z_\tau|\mathcal{F}_n\right) \text{ for each } \tau \in \left(n, T\right), \quad (19)$$

$$J_n = E\left(Z_{\tau_n}|\mathcal{F}_n\right). \quad (20)$$

Taking expectation in (19), we find that $EJ_n \geq E\left(Z_\tau|\mathcal{F}_n\right)$ for all $\tau \in (n,T)$ and hence by taking the supremum over all $\tau \in (n,T)$ we see that $EJ_n \geq V_n$. On the other hand, taking the expectation in (20), we get $EJ_n = E\left(Z_{\tau_n}|\mathcal{F}_n\right)$. Since $\tau_n \in (n,T)$ and (17), it holds that $E\left(Z_{\tau_n}|\mathcal{F}_n\right) \leq V_n$ and therefore, $EJ_n \leq V_n$. The two inequalities give the equality $V_n = EJ_n$, and since $EJ_n = E\left(Z_{\tau_n}|\mathcal{F}_n\right)$, we see $V_n = E\left(Z_{\tau_n}|\mathcal{F}_n\right)$ implying that τ_n is the optimal stopping time to the problem (17).

Appendix D: The Parameter Estimation

In this subsection, the parameters of the diffusion processes, (2), (3) and (6), will be estimated. The Euler-Maruyama method is used to derive the discrete-time approximations of these diffusion processes. For the short term interest rate, the discrete-time approximation is

$$r_{t+\Delta t} - r_t = \kappa_r \left(\overline{r} - r_t\right) \Delta t + u_{r,t+\Delta t}, \tag{21}$$

$$r_{t+\Delta t} = \alpha + \beta_r r_t + u_{r,t+\Delta t}, \tag{22}$$

where the error term, $u_{r,t+\Delta t} = \sigma_r \Delta Z_1$with $\Delta Z_1 = Z_{1,t+\Delta t} - Z_{1,t}$, is normal distributed with

$$\begin{aligned} E_t\left(u_{r,t+\Delta t}\right) &= 0, \\ E_t\left(u^2_{r,t+\Delta t}\right) &= \sigma^2_r \Delta t, \end{aligned}$$

$\alpha = \kappa_r \overline{r} \Delta t$ and $\beta_r = 1 - \kappa_r \Delta t$. The discrete-time approximation of the stock index is

$$S_{t+\Delta t} - S_t = S_t \left(r_t + \lambda_s \sigma_s\right) \Delta t + u_{s,t+\Delta t}, \tag{23}$$

where the error term, $u_{s,t+\Delta t} = \sigma_s S_t \Delta Z_2$ with $\Delta Z_2 = Z_{2,t+\Delta t} - Z_{2,t}$, has the properties

$$\begin{aligned} E_t\left(u_{s,t+1}\right) &= 0 \\ E_t\left(u^2_{s,t+1}\right) &= \sigma^2_s S^2_t \Delta t. \end{aligned}$$

The distribution of the excess return on stock index can be approximated by a normal distribution with mean $\lambda_s \sigma_s \Delta t$ and variance $\sigma^2_s \Delta t$. For this estimation, Δt is taken to be 1, referring to 1 quarter of a year.

The estimation of the AR(1) model (22) is presented in table 10. The AR(1) term of the short rate, β_r, is significant at 1% level. From the estimation reported in table 3, we can get $\kappa_r = 0.0232$ and $\overline{r} = 0.0129$. The volatility of the short rate, σ_r, is derived from the residuals of the two AR(1) process and $\sigma_r = 0.0019$. The price of risk for stock index and the volatility of stock index are estimated from the distribution of excess return of stock index. We have $\lambda_s = 0.1639$ and $\sigma_s = 0.0923$. The correlation coefficient between the two residuals, $u_{r,t+\Delta t}$ and $u_{s,t+\Delta t}$, is -0.1106.

Let the yield of a 10-year zero-coupon government bond derived from Vasicek model be $\hat{Y}_t$, which is a function of λ_r and let Y_t stands for the yield in the data sample. λ_r is estimated by minimizing the objective function, $F\left(\lambda_r\right)$,

$$F\left(\lambda_r\right) = \frac{1}{T} \sum_{t=1}^{T} \left(\hat{Y}_t\left(\lambda_r\right) - Y_t\right)^2.$$

Table 11: Estimation of AR(1) Processes for Stock Index and Short-Term Interest. One star means significance at 10 percent level, two stars mean significance at 5 percent level and three stars mean significance at 1 percent level.

	Short-Term Interest Rates	
α	0.0003	
	(0.0007)	
β_r	0.9768	***
	(0.0319)	
R^2	0.9278	

The price of risk for short-term interest rate, λ_r, is -0.0992.

Chapter 4 Corporate Investment Strategy and Pension Underfunding Risk

This chapter is based on Shi and Werker (2009b).

I Introduction

The recent financial crisis resulted in an unprecedented deterioration of Defined Benefit (DB) pension plans' funding ratios all over the world. Companies sponsoring these underfunded plans are typically required by law to make additional financial contributions to close the funding gap. These days sponsoring companies often have limited ability to obtain outside finance. Additional contributions to pension plans can worsen the financial constraint even further. In this chapter we develop an investment strategy for sponsoring companies aiming to mitigate the impact of liquidity shocks caused by the pension plan underfunding. We also estimate gains of following such a strategy.

With a perfect financial market and no taxes, the Modigliani and Miller (1958) model predicts that additional pension contributions should not affect the investment decision of sponsoring companies. However, Rauh (2006) empirically investigates the impact of DB pension plan underfunding on sponsoring companies' investment decisions in the U.S. He finds strong evidence that sponsoring companies' capital expenditures decline with additional pension contributions. Overall, financial constraints distort the corporate investment in two ways, either through higher capital cost or credit availability. Fazzari, Hubbard and Petersen (1988), Myers and Majluf (1984) and Greenwald, Stiglitz and Weiss (1984), among others, argue that external capital is much more costly than internal capital due to different tax rates, information asymmetry between manager and investor, incomplete contracting and agency costs. Campello, Graham and Harvey (2009) investigate the impact of a credit supply shock on financial policies of financially constrained companies. They survey more than 1000 chief financial officers (CFO) in December 2008. They find direct evidence that a majority of constrained U.S. companies cancel their promising investment projects due to financial constraints.

DB pension plans gained their popularity during and after World War II. In an attempt to control inflation at that time, the wages are binding. Therefore, employers offered generous deferred employee payments for example pension benefits to attract scare labor force (FDIC 2006). Unfavorable financial market performances in recent years increased the cost of DB pension plans and accelerated the trend to shift from DB to DC. However, many companies continue to offer DB pension benefit because they know offering DB pension benefit gives them better chances in hiring and retaining the best employees (FDIC 2006). DB pension benefit is often defined as a certainty percentage of salaries. Salaries of a worker close to retirement is typically higher than the ones in the mid of his career. Thus, if he leaves the employer early, he will get less pension. At this moment, there are still 34 million working and retired Americans depend on DB pension plans (FDIC 2006). The healthiness of the sponsoring companies is vital to support these DB pension plans.

In this chapter, we find an optimal dynamic investment strategy which minimizes the impact of DB pension plan underfunding on the value of the sponsoring company. In our model, the company has a limited amount of internal capital, an opportunity to invest in a project within a certain period and a limited credit supply. The company sponsors a DB pension plan and has an obligation to make additional financial contributions. We show that the company's optimal investment strategy depends on the amount of capital available for investment and the initial pension funding ratio unless the company is extremely financially constrained or unconstrained. The amount of capital available for investment is the sum the internal capital and the amount of capital borrowed less pension contributions. The results indicate that firms with lower pension funding ratio should have a lower investment threshold value than otherwise identical firms with higher pension funding ratios. The investment threshold value is the lowest project value above which the firm will invest. The result is driven by the fact that lower pension funding ratios mean higher expected future pension contributions and therefore lower values of waiting: the risk that in the future the capital available for investment will be used to fill the pension funding gap decreases the value of waiting.

We compare values of two otherwise identical firms following two different strategies. One firm follows the proposed investment strategy taking into account the pension underfunding risk while the other firm follows an investment strategy ignoring the pension underfunding risk. We find that following the optimal investment strategy can increase the investment option value significantly.

Optimal corporate investment strategies have been investigated extensively in the real option literature. McDonald and Siegel (1986) study the corporate investment timing decision for an investment which is irreversible but can be delayed. Standard real option approaches to corporate investment strategies are explained in detail in Dixit and Pindyck (1994). Boyle and Guthrie (2003) examine the dynamic investment decision of a firm subject to a financial constraint with an infinite investment horizon. They show that a liquidity-constraint company can have a much lower investment threshold value than an otherwise identical company without a liquidity constraint due to a lower value of waiting. As the amount of capital available for investment increases, the investment threshold shifts up and eventually converges to the unconstrained one. This chapter extends the Boyle and Guthrie (2003) paper by taking into account the interaction between the firm's investment strategy, the firm's liquidity constraint and underfunding risk of the sponsored pension plan. These two papers also differ in the length of investment option life. In Boyle and Guthrie (2003), the investment option has an infinite life, while in our model the investment option has a finite life, since competition and innovation in a fast developing knowledge economy as today's very often limit the life of an investment option.

Webb (2007) studies the interaction between a DB pension plan sponsoring company's capital structure, dividend policy, investment strategy and pension contribution decision with and without the presence of Pension Benefit Guarantee Corporation (PBGC). This chapter extends Webb (2007) by considering a financial constraint and a dynamic investment timing decision.

Borrowing constraints and lower investment returns can cause pension plan underfunding (Cooper and Ross 2002). Underfunding can also be caused by employers' intensions to strengthen their bargaining power against labour unions (Ippolito 1985a, b), and to share risk with employees (Arnott and Gersovitz 1980). No matter what is the cause of underfunding, nowadays, firms all over the world are required to make additional contribution to their underfunded pension plans to fill the gap. In this chapter, we focus on the impact of additional pension contributions on the liquidity condition and the investment decision of the sponsoring firm, while ignoring the source of underfunding.

The outline of this chapter is as follows, Section 2 gives an overview of the model including the asset and liability structure of the firm and its investment environment. Section 3 describes the optimal investment strategy under pension underfunding risk and the increase in the investment project value by following such a strategy. Section 4 concludes.

II The Investment Environment

At time 0, a fully equity financed firm has, among others, an investment option, a limited amount of internal capital, a DB pension plan and some illiquid assets. The firm can exercise the investment option at any time between time 0 and time T. If the option is exercised at time t, $t \in [0, T]$, the firm pays a fixed amount I and receives a project V_t.

In a risk-neutral world, the project value follows a diffusion process

$$dV_t = (r - \delta_v) V_t dt + \sigma_v V_t dW_{v,t}, \tag{1}$$

where r is a risk-free investment rate, δ_v is the "convenience yield" of the project[1], σ_v is the volatility of the project value and $dW_{v,t}$ is a standard Brownian motion under the risk neutral probability.

The amount of internal capital follows the diffusion process

$$dX_t = rX_t dt + \sigma_x X_t dW_{x,t}, \tag{2}$$

where σ_x is the volatility of the internal capital and $dW_{x,t}$ is a standard Brownian motion.

The value of the firm's pension liability is assumed to be constant over time at value L. The diffusion process of the funding ratio[2] is

$$dF_t = (r - \delta_f) F_t dt + \sigma_f F_t dW_{f,t}, \tag{3}$$

where δ_f is the cost of the pension fund and $dW_{f,t}$ is a standard Brownian motion. The correlations between the three Brownian motions are $\rho_{v,x}$, $\rho_{v,f}$ and $\rho_{x,f}$ respectively.

If at time T the funding ratio is smaller than 1, the firm has to make additional contributions to make the pension plan fully funded again. The sponsoring company is not allowed to withdraw surplus from the pension plan[3]. Let A_T be the firm's pension contribution and

$$A_T \equiv (1 - F_T) L \mathbf{1}_{F_T<1},$$

where $\mathbf{1}_{F_T<1}$ is an indicator function. Because the funding ratio is log normally distributed, the expected pension contribution, $E_t A_T$, is

$$E_t A_T = \Phi(d_1) L \Phi(d_2) - F_t L \exp((r - \delta_f)(T - t)), \ \forall t < T,$$

[1] The "convenience yield" can be the marginal benefit generated by the project, for example, an increased ability to smooth production. See Dixit and Pindyck (1994) for a more detailed description of the marginal benefit.

[2] The pension funds are subject to various types of regulation, for example, value-at-risk. The impacts of these regulations on the pension funds' funding ratios are out of the scope of this chapter.

[3] We are focusing on extremely underfunded pension plans. For these pension plans, it is very unlikely that they will be overfunded in a very short horizon, for example one year, and, therefore, the surplus withdrawal will not play a role here.

where $\Phi(\cdot)$ stands for the standard normal cumulative distribution function,

$$d_1 = \frac{-\log F_t - \left(r - \delta_f + \frac{1}{2}\sigma_f^2\right)(T-t)}{\sigma_f\sqrt{(T-t)}}$$

and

$$d_2 = \frac{-\log F_t - \left(r - \delta_f - \frac{1}{2}\sigma_f^2\right)(T-t)}{\sigma_f\sqrt{(T-t)}}.$$

The pension contribution is the firm's debt. The time horizon that this chapter considers is very short (at the most one year). Usually the firms will not be required to contribute to the pension plan within a year. Therefore, we only consider underfunding at time T.

The firm can raise new funds to finance the investment project at any time. But the capacity to raise new funds is limited due to various reasons, for example, moral hazard and agency issues. As in Boyle and Guthrie (2003), we assume that the amount the firm can borrow is a certain percentage of the project value at the time of borrowing. Let c be the borrowing limit with $0 \leq c < 1$. c measures the capital market friction and is assumed to be exogenous. The amount of cash available for investment, B_t, consists of the operating asset, X_t, and the amount borrowed, cV_t, before time T. At time T, cash includes the operating asset, X_T, the amount borrowed, cV_T, less the additional pension contribution, A_T

$$B_t = \begin{cases} X_t + cV_t & \text{for } t < T \\ X_T + cV_T - A_T & \text{for } t = T \end{cases}. \tag{4}$$

At time t, the firm can invest in the project if and only if $B_t \geq I$. That means the project value should be at least as large as $(I - X_t)/c$ before time T and $(I - X_T + A_T)/c$ at time T, otherwise, the firm will not have enough capital to finance the investment. The second equation in (4) indicates that the firm should first pay pension contribution if he decides to invest in the project at time T.

Table 1 presents the balance sheet of the firm before and immediately after the investment takes place. Assume that the investment takes place at time i. The equity value before the investment takes place (M_t) is the sum of option value (Q_t), other assets (OA_t) and cash (X_t) less the expected pension contribution (E_tA_T), $t < i$. We assume immediately after investment takes place, the firm will cash in the gain and repay the debt. The equity value after the investment takes place is sum the cash ($V_i - I + X_i$) and other asset value (OA_t) less the expected pension contribution (E_tA_T), $T >= t >= i$.

Table 2 summarizes the timeline of the investment decision and pension contribution requirement. The investment option will expire after time T. Before time T, the firm has

Table 1: This table shows the liquidity constrained firm's balance sheet.

Before Investment Takes Place ($t < i$)			
Option Value	Q_t	Value of Debts	0
Cash	X_t	Expected Pension contribution	$E_t A_T$
Illiquid Assets	OA_t		
		Value of Equity	M_t
Total	$Q_t + X_t + OA_t$		$M_t + E_t A_T$
Immediately After Investment Takes Place ($T \geq t \geq i$)			
Cash	$V_t + X_t - I$	Value of Debts	0
Illiquid Assets	OA_t	Expected Pension contribution	$E_t A_T$
		Value of Equity	M_i
Total	$V_i - I + X_i + OA_t$		$M_t + E_t A_T$

a number of chances to decide whether to invest immediately or to wait. When the firm decides to invest, he will borrow from the financial market if he is liquidity constrained. At time T, the firm has to make contributions to the pension plan if the pension plan is underfunded. If the firm decides to invest in the project, he has to make the pension contribution before he invests.

The firm is assumed to be fully equity financed and controlled by the shareholders. In line with their goal to maximize the firm value, the shareholders want to find the optimal investment time which maximizes the value of the investment option at time 0, Q_0. That is, the firm is expected to invest at time τ^* such that the expected value at time 0 is maximized[4]:

$$Q_0\left(\tau^*\right) = E_0\left(V_{\tau^*} - I\right)\exp\left(-r\tau^*\right), \tag{5}$$

where $Q_0\left(\tau^*\right) \geq Q_0\left(\tau\right)$, for all $\tau \in (0, T)$, and the borrowing constraint is fulfilled, i.e., $B_{\tau^*} \geq I$, where B_{τ^*} is the amount of capital available for investment which is explained in (4). The option value at time 0, $Q_0\left(\tau^*\right)$, is the discounted investment gain, $V_{\tau^*} - I$. We will present the option value and the numerical solution for the optimal investment

[4]It is required by law that a firm should contribute capital to its underfunded pension plan. There is no regulation about the timing of the investment.

Table 2: This table shows the timeline of the investment decision and pension contribution requirement.

Before Time T	**At Time T**
The firm has a number of chances to decide whether to exercise the investment option immediately or to wait.	If the pension fund is underfunded, the firm must pay additional contributions to fill the gap.
The firm only borrow at the time when he decides to invest.	Exercising the investment option, if there is enough cash and the project is profitable.
The investment decision will depend on the amount of capital available.	If the investment hasn't been occured by then, the option expires.

timing problem in next sections.

III The Optimal Investment Strategy and the Option Value

The Numerical Solution

The corporate investment strategy under pension underfunding risk resembles the optimal strategy of exercising an American option with finite horizon. We adopt the algorithm developed by Ibáñez and Zapatero (2004) to solve this problem numerically. The algorithm is originally designed to estimate the value of an American option using the dynamic programming principle.

At time T, without a liquidity constraint, if the firm decides to invest, the option value is

$$\max\left(V_T - I, 0\right).$$

Without the liquidity constraint, the exercise frontier at time T is the investment cost I. As soon as the project value is larger than the threshold value, the firm will invest. The liquidity constraint, $X_T + cV_T - A_T \geq I$, indicates that the firm will have enough cash to undertake the project if and only if the project value is not smaller than $\frac{I-X_T+A_T}{c}$. To fulfill both conditions, the exercise frontier at the final exercise time T, $V_T^*(X_T, A_T)$, is

$$V_T^*(X_T, A_T) = \max\left(\frac{I - X_T + A_T}{c}, I\right).$$

At time $T - \Delta t$, the value of immediate exercise is the investment gain $\max(V_{T-\Delta t} - I, 0)$ and the value of waiting is the discounted expected value of exercising the option optimally at time T, $\eta_{T-\Delta t}$, where

$$\eta_{T-\Delta t} = E_{T-\Delta t}(V_T - I)\, 1_{V_T \geq V_T^*(X_T, A_T)}.$$

Without the liquidity constraint, the firm will exercise the investment option as soon as the value of immediate exercise is larger than the value of waiting

$$V_{T-\Delta t} \geq \eta_{T-\Delta t} + I. \tag{6}$$

The presence of a liquidity constraint means that the amount of cash available for investment $X_{T-\Delta t} + cV_{T-\Delta t}$ should be larger than the investment cost I, that gives

$$V_{T-\Delta t} \geq \frac{I - X_{T-\Delta t}}{c}. \tag{7}$$

For any given values of $X_{T-\Delta t}$ and $F_{T-\Delta t}$, the early exercise frontier is the maximum of $\eta_{T-\Delta t} + I$ and $\frac{I-X_{T-\Delta t}}{c}$.

For any given values of $X_{T-\Delta t}$ and $F_{T-\Delta t}$, the value of immediate exercise, $V_{T-\Delta t} - I$, is an increasing and linear function of the project value, $V_{T-\Delta t}$, while the value of waiting, $\eta_{T-\Delta t}$, is an increasing and convex function of the project value, $V_{T-\Delta t}$, which is a common property of American options. This property guarantees that for any given values of $X_{T-\Delta t}$ and $F_{T-\Delta t}$ there will be a value $V_{T-\Delta t}^*$ such that it is optimal to exercise the option at time $T - \Delta t$ as soon as the project value exceeds the threshold value, $V_{T-\Delta t}^*$. The details of the procedure to derive the investment threshold value are provided in the appendix. The main idea proceeds as follows. At time $T - \Delta t$, we first take a grid of points, $(\mathbf{X}_{T-\Delta t}, \mathbf{F}_{T-\Delta t})$, where $\mathbf{X}_{T-\Delta t}$ and $\mathbf{F}_{T-\Delta t}$ are two vectors consisting of simulated liquid asset values and funding ratios. Then we compute the corresponding investment threshold values, $\mathbf{V}_{T-\Delta t}^*$, where $\mathbf{V}_{T-\Delta t}^*$ is a vector consisting of investment threshold values corresponding to the vectors $\mathbf{X}_{T-\Delta t}$ and $\mathbf{F}_{T-\Delta t}$. Afterwards, we regress the threshold values, $\mathbf{V}_{T-\Delta t}^*$, on polynomials of $\mathbf{X}_{T-\Delta t}$ and $\mathbf{F}_{T-\Delta t}$ to allow for a parameterization of the early exercise frontier. Proceeding backwards, we use the same method to parameterize the exercise frontiers at exercise times before $T - \Delta t$.

After parameterizing the exercise frontiers at each possible exercise time, we simulate other N paths for cash stocks, X, project values, V, and funding ratios, F, from time 1 to time T. For each path, the optimal time to invest is the first time the value of the project is not smaller than the early exercise frontier. Let τ_i^* denote the optimal investment time for path i, $i = 1, 2, ..., N$. Let H be a $N \times T$ matrix, where the rows correspond to the simulated paths and the columns correspond to time. The matrix H records the optimal investment decisions of the firm. If $H(i, j) = 1$, j is the optimal investment time for path i, otherwise, j is not the optimal investment time for path i

$$H_{i,j} = \begin{cases} 1 & \text{if } j = \tau_i^* \\ 0 & \text{otherwise} \end{cases} . \tag{8}$$

By construction, there will be at the most only one "1" in each row. Let M be the matrix recording the optimal payoff at each path and each time. The value of $M_{i,j}$, is a function of the decision rule, $H_{i,j}$ and the gain from investment, $V_{i,j} - I$,

$$M_{i,j} = [H_{i,j} \times (V_{i,j} - I)] . \tag{9}$$

At time 0, the investment option value is

$$Q_0 = \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{T} M_{i,j} \exp(-rj) ,$$

which is the average discounted value of the optimal payoffs over N simulated paths.

The Optimal Investment Strategy

In this section, we will show investment strategies for firms with various pension funding ratios. We assume that the investment option will expire in one year and within this year the firm has four times to exercise the investment option. Table 3 gives the baseline parameter values used for the numerical solution procedure.

Optimal Investment Strategy Without Pension Plans

Figure 1 shows the early exercise frontiers for liquidity constrained firms without pension plans. For a liquidity constrained firm, the early exercise frontier depends on the time to maturity and the value of liquid assets. At any time before maturity, the relationship

Table 3: Baseline Parameter Values Used in the Numerical Solution Procedure

Parameter	**Values**
Project Investment Cost (I)	20
Interest Rate (r)	0.05
Initial Value of the Project (V_0)	20
Volatility of the Project Value (σ_v)	0.20
Volatility of Liquid Asset Value (σ_x)	0.20
Volatility of Funding Ratio (σ_f)	0.20
Value of Pension Liability (L)	25
Time to Maturity (T)	1
Market Friction (c)	0.25
Correlation between Liquid Asset and Project Value (ρ_{xv})	0
Correlation between Funding Ratio and Project Value (ρ_{fv})	0
Convenience Yield of Investment (δ_v)	0.03
Pension Surplus Sharing Rule (α)	0
Pension Fund Management Cost (δ_f)	0.03

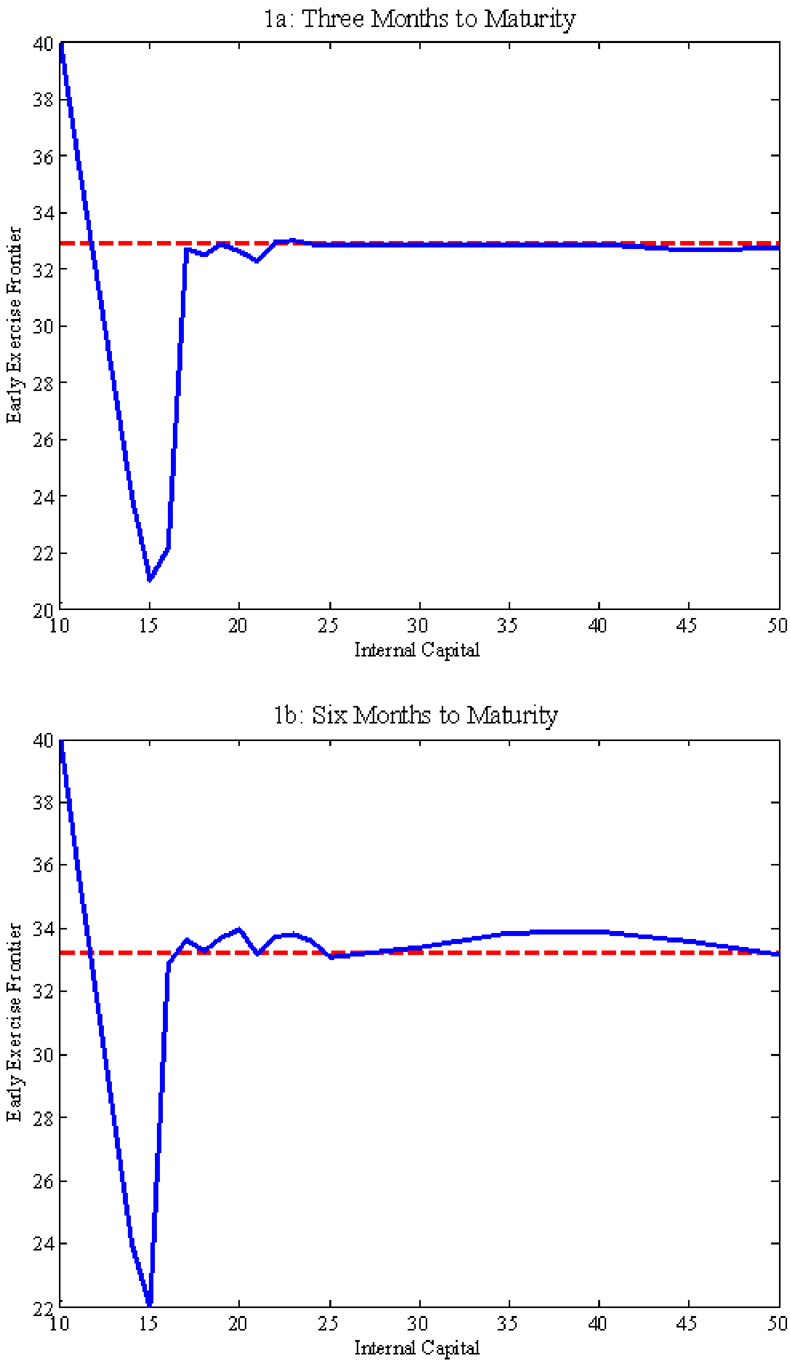
between the exercise frontier and the value of liquid assets has a "V" shape. When the value of liquid assets is low, the early exercise frontier is linearly decreasing with the value of liquid assets. At low levels of liquid asset value, the investment strategy is dominated by the borrowing constraint. As the value of liquid assets increases, both the value of immediate exercise and the value of waiting increase but the latter increases faster than the former. Therefore, the early exercise frontier shifts upwards. As the value of liquid assets increases further, the firm becomes liquidity unconstrained and the early exercise frontier converges to the unconstrained one.

Optimal Investment Strategy With Pension Plans

Figure 2 shows early exercise frontiers for different pension funding ratios at three months to maturity. Firms facing different pension funding ratios should follow different investment timing strategies, even though these are firms who are otherwise completely identical, unless they are extremely financially constrained or unconstrained. At three month to maturity, when the value of liquid assets is less than 75% of investment cost, i.e. $X_0 \leq 15$, the early exercise frontier is determined by the borrowing constraint. At this level of liquid assets value, the investment strategies of companies with different pension funding ratios are the same. As the value of liquid assets increases, the early exercise frontier shifts up for all pension funding ratios. But the shift is less pronounced for companies with lower funding ratios. This is due to the fact that a larger expected pension deficit makes it much more risky to wait. Thus, when the value of liquid assets is between 75% and 150% of investment cost, i.e., $15 < X_0 \leq 30$, the investment threshold increases with the pension funding ratio. As the value of liquid assets increases further, all firms become unconstrained and therefore have the same investment strategies again. The early exercise frontiers at earlier times have similar shapes as the ones at three months to maturity, except that they are converging to a higher level because the value of waiting is getting smaller as it gets closer to expiration.

The Investment Option Value

The value of investment project at time 0 for firms without pension plans is presented in table 4. For a unconstrained firm, the investment option value, Q_0, is 1.75, which is independent of the amount of the internal capital. While for an otherwise identically liquidity constrained firm, the investment option value depends heavily on the amount of internal capital. When the asset value is 10, which is about 50% of investment cost


1a: Three Months to Maturity
Early Exercise Frontier
Internal Capital
40
38
36
34
32
30
28
26
24
22
20
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15
20
25
30
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45
50
1b: Six Months to Maturity
Early Exercise Frontier
Internal Capital
40
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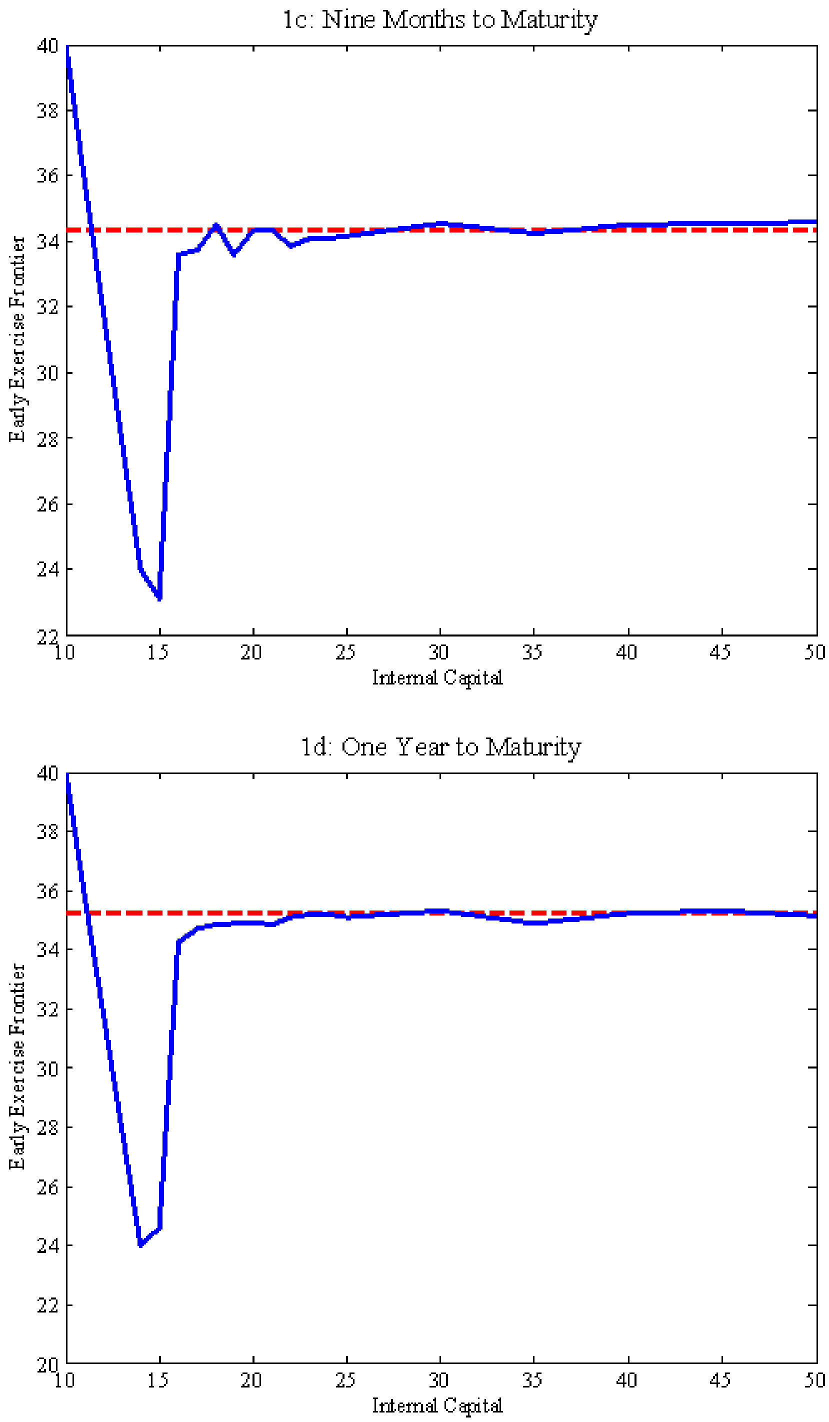

Figure 1: This figure shows the early exercise frontiers of the investment option at three months, six months, nine months and one year to maturity respectively in the absence of a pension plan. The early exercise frontiers determined by $\max(I+\eta_t,\ (I-X_t)/c)$, $t \in [0,T]$, where η_t is the value of waiting at time t, I is the investment cost, X_t is the amount of internal capital and c is borrowing constraint which means the maximum amount the firm can borrow is cV_t. The red lines are the frontiers for an unconstrained firm and the blue dashed lines are the frontiers for an otherwise identical liquidity constrained firm. The parameter values used for simulation are presented in table 1.

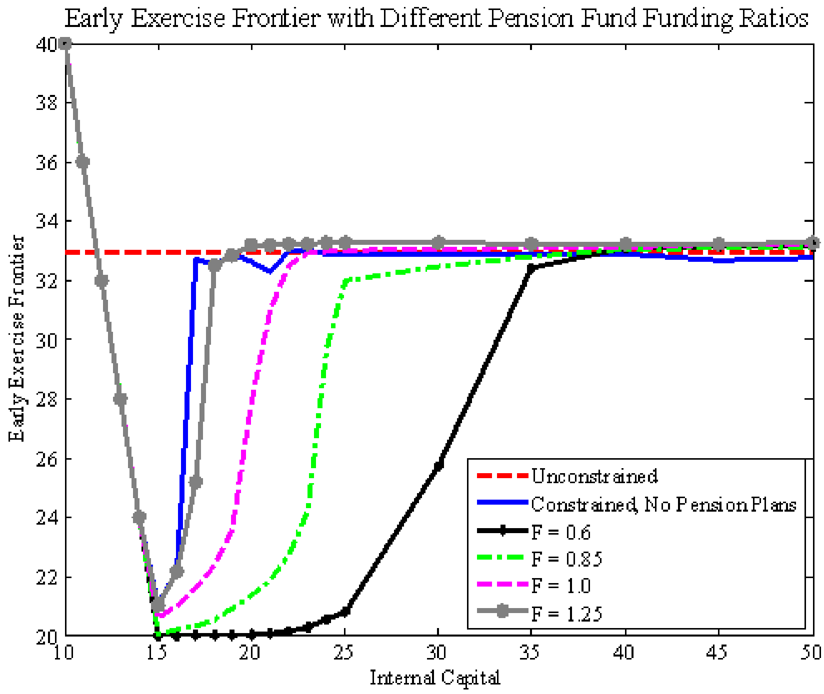

Figure 2: This figure shows the early exercise frontiers of the investment option for a liquidity constrained firm sponsoring a pension plan at three months to maturity. The pension fund funding ratios are 0.6, 0.85, 1.0 and 1.25 respectively.

(I), the option value is 0.23, which means that the liquidity constraint reduces the option value by 87%. When the amount of internal capital is 15, which is 75% of the investment cost, I, the option value is 1.4 and the reduction in option value is 21%. The option value of a liquidity constrained firm converges to the one of a unconstrained firm as the amount of internal capital increases. Table 4 also shows how likely that the investment option will be exercised. Without the liquidity constraint, there is 50% of probability that the firm will invest within one year. The investment probability is reduced dramatically by the liquidity constraint. When the amount of internal capital is 10, there is only 4.4% of probability that the firm will invest. When the amount of internal capital is 15, there is 39% of probability that the firm will invest.

The loss an underfunded pension plan brings to the sponsoring firm not only consists of the amount of mandatory contributions but also of the reduced ability to invest in its core business due to the liquidity constraint caused by the pension underfunding. Figure 2 proposes an investment timing strategy for the firm with underfunded pension plan which aims to maximize the option value given pension underfunding. Table 5 and 6 presents investment option values of two identical firms following different investment strategies with the initial pension funding ratio equals to 0.6, and 0.85 respectively. Both firms are

Table 4: This table shows the investment option values and investment probabilities for firms without pension plans. The parameter values are presented in table 1. X_0 stands for the amount of internal capital.

	Option Value			Option Exercise Percentage	
X_0	Unconstrained	Constrained	% Reduction	Unconstrained	Constrained
10	1.7516	0.2305	−86.84%	50.00%	4.38%
11	1.7516	0.4212	−75.95%	50.00%	9.18%
12	1.7516	0.6850	−60.89%	50.00%	16.30%
13	1.7516	0.9560	−45.42%	50.00%	24.90%
14	1.7516	1.2058	−31.16%	50.00%	33.11%
15	1.7516	1.3888	−20.71%	50.00%	39.04%
16	1.7516	1.5296	−12.67%	50.00%	43.54%
17	1.7516	1.6184	−7.60%	50.00%	46.02%
18	1.7516	1.6696	−4.68%	50.00%	47.75%
19	1.7516	1.7135	−2.18%	50.00%	48.61%
20	1.7516	1.7289	−1.30%	50.00%	49.16%
21	1.7516	1.6810	−4.03%	50.00%	49.44%
22	1.7516	1.6967	−3.13%	50.00%	49.76%
23	1.7516	1.7240	−1.58%	50.00%	49.85%
24	1.7516	1.7329	−1.07%	50.00%	49.86%
25	1.7516	1.7356	−0.91%	50.00%	49.95%
26	1.7516	1.7476	−0.23%	50.00%	49.98%
27	1.7516	1.7216	−1.71%	50.00%	49.98%
28	1.7516	1.7434	−0.47%	50.00%	50.00%
29	1.7516	1.7406	−0.63%	50.00%	50.00%
30	1.7516	1.7248	−1.53%	50.00%	50.00%
31	1.7516	1.7217	−1.71%	50.00%	50.00%
32	1.7516	1.7400	−0.66%	50.00%	50.00%
33	1.7516	1.7334	−1.04%	50.00%	50.00%
34	1.7516	1.7338	−1.02%	50.00%	50.00%
35	1.7516	1.7290	−1.29%	50.00%	50.00%
36	1.7516	1.7416	−0.57%	50.00%	50.00%
37	1.7516	1.7256	−1.48%	50.00%	50.00%
38	1.7516	1.7128	−2.22%	50.00%	50.00%
39	1.7516	1.7297	−1.25%	50.00%	50.00%
40	1.7516	1.7275	−1.38%	50.00%	50.00%

sponsoring pension plans facing the same pension plan funding ratios. Firm 2 follows the optimal investment strategy taking into account the pension underfunding risk and firm 1 follows the strategy which is optimal for a liquidity-constrained firm without a pension plan.

For a financially constrained firm with underfunded pension plans, the pension contribution can lead to a large drop of the investment option value. For a firm sponsoring a 60% funded pension plan, even if they follow the optimal strategy taking into account the pension plan underfunding risk (firm 2 in table 5), the pension contribution can reduce the option value from 0.23 (1.38) to 0.10 (1.1) when the amount of internal capital is 10 (15). For an otherwise identical firm sponsoring a 85% funded pension plan, the pension contribution can reduce the option value from 0.23 (1.38) to 0.12 (1.26) when the amount of internal capital is 10 (15).

Results presented in table 5 and table 6 also show that the investment strategy matters a lot for a firm who is neither extremely constrained nor unconstrained. The benefit of following the strategy taking into account the pension fund underfunding risk ("% Increase" in table 5 and 6) is first increasing with the amount of internal capital and then decreasing. The same conclusions hold for both 60% and 85% funded pension plans. When the initial value of liquid assets is low, the investment frontier is determined by the borrowing constraint and therefore the firm values do not differ a lot. As the value of liquid assets increases to 20, the differences in option value increase to about 78% for firms with 60% funded pension plan and 28% for firms with 85% funded pension plans[5]. For a unconstrained firm, the investment strategy will not be affected by the pension fund underfunding. Therefore, the benefit is negligible. Since the firm repays their debt immediately after the investment takes place, higher option value means higher equity value. The investment strategy proposed by this chapter leads to a higher option value and thus, also a higher equity value.

Table 5 and 6 also presented the probabilities that the investment will take place. For a firm with the amount of internal capital smaller than 15, the lower the pension funding ratio, the less likely that the investment will take place, no matter which strategy the firm follow. However, a firm facing 60% funded pension plan and owning an amount of internal capital between 15 and 27 is more likely to exercise the investment option than the unconstrained firm if the firm follows the investment strategy taking into account the pension plan underfunding risk. As can be seen from figure 2, a firm with underfunded

[5] The increase in option value is due to the difference in investment strategy. Firm 1 and firm 2 are identical firms. The only difference between these two firms is the investment strategy.

pension plan should have a lower exercise frontier than the financially unconstrained one unless the firm is extremely financially constrained or rich in cash. The only difference between strategies followed by firm1 and firm 2 lie in the option exercise timing. The gains of following the strategy taking into account the pension plan underfunding risk shows that timing the investment promptly is important.

Robustness Check

In this section, we will investigate the investment strategies and the gains of following the investment strategy taking into account the pension plan underfunding risk with different parameter values. The results for funding ratio of 0.6 are presented. The results for other pension funding ratios which are smaller than 1 are very similar.

Figure 3 shows the investment strategy with different volatilities of the amount of cash owned by the firm, σ_x. The early exercise frontier is decreasing with σ_x, unless the firm is extremely financially constrained or unconstrained. Higher σ_x means higher probability that the firm will have less cash for investment in the future. Therefore, the firm with higher σ_x will invest earlier than otherwise identical firm with lower σ_x. Table 7 shows gains of following the strategy taking into account the pension plan underfunding risk with different σ_x. As in the previous section, the gain is measured by the percentage increase in the investment option in response to follow the strategy taking into account the pension plan underfunding risk. This strategy can bring a large increase in project value for all the values of σ_x unless the firm is extremely constrained or unconstrained. For example, for a firm with cash equals to the investment cost, $I = 20$, the gains of following the strategy proposed by this section are 147%, 72% and 25% for a σ_x equals to 10%, 20% and 30% respectively.

Figure 4 shows the investment strategies with different pension fund funding ratio volatilities, σ_f. The larger the funding ratio volatility, the more likely that there will be larger pension contribution in the future[6]. Therefore, firms with larger pension funding ratio volatility will have a lower investment threshold value than otherwise identical firms with a lower funding ratio volatility. The gains of following the strategy proposed is shown in table 8. The higher the pension funding ratio volatility, the larger the gain. For

[6] At time T, if the funding ratio is below 1, the firm has to contribute to make it fully funded again. If funding ratio is above 1, the firm will not share the surplus. It is also very unlikely a 60% funded pension plan will be well funded in one year with realistic funding ratio volatility value.

Table 5: This table shows the investment option values and investment probabilities for firms sponsoring a pension fund which is 60 percent funded. Firm 2 follows the optimal investment strategy taking into account the pension underfunding risk and firm 1 follows the strategy which is optimal for a liquidity-constrained firm without a pension plan. The parameter values are presented in table 1. X_0 stands for the amount of internal capital.

Cash	Option Value			Exercise Probabilities	
X_0	Firm 1	Firm 2	% Increase	Firm 1	Firm 2
10	0.0986	0.1011	2.58%	1.76%	2.22%
11	0.2493	0.2640	5.90%	4.84%	6.62%
12	0.4589	0.5030	9.61%	9.76%	14.98%
13	0.6498	0.7271	11.90%	14.70%	25.02%
14	0.7903	0.9466	19.78%	18.51%	36.44%
15	0.8740	1.1010	25.98%	20.38%	46.62%
16	0.8139	1.2180	49.65%	18.91%	54.18%
17	0.7816	1.2331	57.76%	17.84%	59.00%
18	0.7170	1.2755	77.89%	16.74%	62.16%
19	0.7616	1.3079	71.74%	17.66%	63.68%
20	0.7481	1.3042	74.32%	18.20%	63.70%
21	0.8161	1.3375	63.89%	20.25%	62.15%
22	0.9154	1.3738	50.08%	23.04%	60.44%
23	0.9646	1.4465	49.97%	25.44%	58.76%
24	1.0858	1.4597	34.44%	28.98%	56.28%
25	1.1771	1.5474	31.47%	31.76%	54.46%
26	1.2978	1.5504	19.46%	35.48%	52.75%
27	1.3112	1.5952	21.66%	37.32%	51.90%
28	1.4499	1.6583	14.37%	40.10%	50.82%
29	1.4811	1.6147	9.02%	41.76%	50.14%
30	1.5293	1.6600	8.55%	43.47%	49.77%
31	1.5628	1.6559	5.96%	44.90%	49.49%
32	1.6257	1.6894	3.92%	45.94%	49.32%
33	1.6540	1.7078	3.25%	46.98%	49.54%
34	1.6788	1.6917	0.77%	47.70%	49.38%
35	1.6822	1.7113	1.73%	48.19%	49.46%
36	1.6710	1.7167	2.73%	48.68%	49.44%
37	1.7187	1.7223	0.21%	48.92%	49.60%
38	1.7045	1.7188	0.84%	49.28%	49.68%
39	1.7337	1.7028	−1.78%	49.44%	49.78%
40	1.7154	1.7245	0.53%	49.64%	49.91%

Table 6: This table shows the investment option values and investment probabilities for firms sponsoring a pension fund which is 85 percent funded. Firm 2 follows the optimal investment strategy taking into account the pension underfunding risk and firm 1 follows the strategy which is optimal for a liquidity-constrained firm without a pension plan. The parameter values are presented in table 1. X_0 stands for the amount of internal capital.

	Option Value			Exercise Probability	
	Firm 1	Firm 2	% Increase	Firm 1	Firm 2
10	0.1171	0.1217	3.88%	2.26%	2.46%
11	0.2887	0.3003	4.03%	5.87%	6.73%
12	0.5063	0.5326	5.21%	10.92%	13.78%
13	0.7188	0.8059	12.12%	17.11%	22.76%
14	0.9576	1.0563	10.31%	23.60%	31.36%
15	1.0544	1.2642	19.90%	26.50%	38.28%
16	1.1661	1.3813	18.46%	28.93%	42.72%
17	1.1358	1.4577	28.34%	29.86%	45.80%
18	1.1965	1.5089	26.12%	30.92%	47.09%
19	1.2375	1.5295	23.59%	32.86%	48.70%
20	1.3022	1.5868	21.86%	34.89%	48.60%
21	1.3698	1.6004	16.84%	36.70%	49.06%
22	1.4108	1.6330	15.75%	38.70%	49.31%
23	1.4628	1.6481	12.67%	40.99%	49.20%
24	1.5214	1.6669	9.56%	42.62%	49.52%
25	1.5745	1.6697	6.05%	43.94%	49.37%
26	1.6311	1.6878	3.48%	45.50%	49.36%
27	1.6318	1.6827	3.12%	46.44%	49.46%
28	1.6789	1.7119	1.97%	47.13%	49.58%
29	1.6695	1.7011	1.89%	47.92%	49.56%
30	1.6983	1.7099	0.68%	48.47%	49.62%
31	1.7154	1.6970	−1.07%	48.92%	49.66%
32	1.7059	1.6799	−1.52%	49.24%	49.68%
33	1.7353	1.7266	−0.50%	49.50%	49.84%
34	1.7177	1.7414	1.38%	49.51%	49.84%
35	1.7165	1.7135	−0.18%	49.76%	49.88%
36	1.7176	1.7457	1.64%	49.78%	49.94%
37	1.7448	1.7410	−0.22%	49.84%	49.96%
38	1.7209	1.7387	1.04%	49.89%	49.98%
39	1.7235	1.7160	−0.43%	49.94%	49.97%
40	1.7053	1.7213	0.94%	49.95%	49.98%

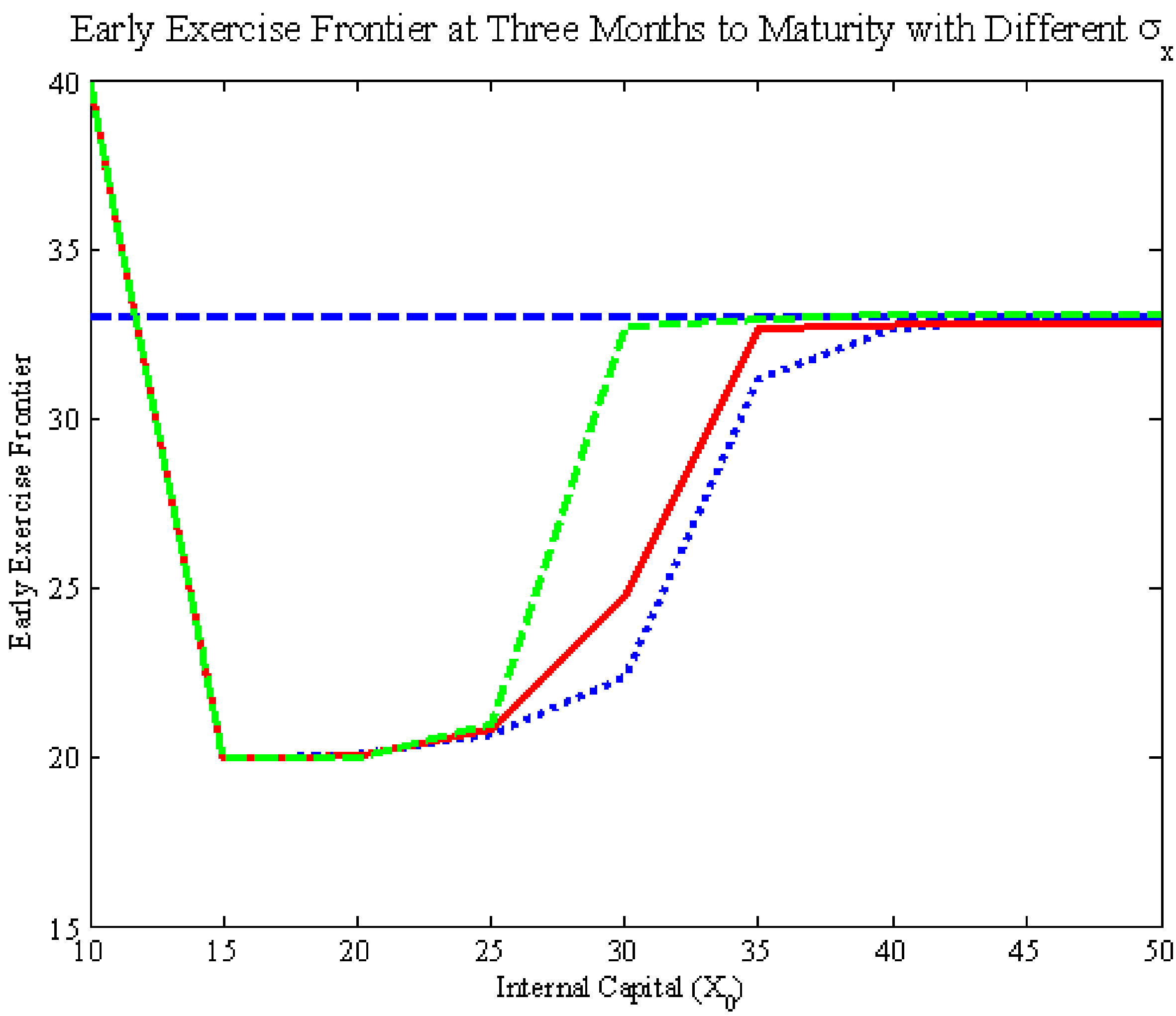

Figure 3: This figure shows the early exercise frontier at three months to maturity with different volatility of cash, σ_x. The pension funding ratio is 0.6. The horizontal dashed line is the frontier for a unconstrained firm. The dashed curve is the frontier for a constrained firm with the volatility of cash, σ_x, equals to 0.1. The red solid curve is for $\sigma_x = 0.2$. The blue dotted curve is for $\sigma_x = 0.3$.

Table 7: This table shows gains of following a strategy taking into account of the penison plan underfunding risk with different cash volatilities, σ_x. The gain is defined as the percentage difference between of two options values: one follows the investment strategy taking into account the pension plan underfunding risk and the other one following the strategy ignoring the pension plan underfunding risk. Other parameter values are presented in table 1.

X_0	$\sigma_x = 0.3$	$\sigma_x = 0.2$	$\sigma_x = 0.1$
10	0.82%	2.58%	8.6%
15	12.45%	25.98%	103.6%
16	17.96%	49.65%	223.9%
17	18.67%	57.76%	373.7%
18	22.09%	77.89%	317.7%
19	23.62%	71.74%	212.7%
20	24.99%	74.32%	147.4%
40	0.63%	0.53%	−1.1%

Table 8: This table shows gains of following a strategy taking into account of the penison plan underfunding risk with different pension plan funding ratio volatilities, σ_f. The gain is defined as the percentage difference between of two options values: one follows the investment strategy taking into account the pension plan underfunding risk and the other one following the strategy ignoring the pension plan underfunding risk. Other parameter values are presented in table 1.

	$\sigma_f = 0.3$	$\sigma_f = 0.2$	$\sigma_f = 0.1$
10	6.69%	2.58%	−3.17%
15	39.26%	25.98%	25.82%
16	54.56%	49.65%	42.15%
17	75.02%	57.76%	59.42%
18	88.37%	77.89%	70.24%
19	87.01%	71.74%	74.93%
20	79.82%	74.32%	78.53%
40	0.95%	0.53%	−0.68%

example, when the amount of cash the firm has is 18, the gain is 88%, 78% and 70% if the funding ratio volatilities are 30%, 20% and 10% respectively.

Figure 5 shows the early exercise frontier with different interest rates. The higher the interest rate, the higher the expected return from both the investment project and the pension fund. Therefore, higher interest rate leads to higher value of waiting and higher early exercise threshold value. As can be seen from table 9, the strategy proposed by this chapter generate higher gains for interest rates equals to 5% and 7% compared with those for interest rate equal to 3%. For example, when the amount of cash is 15, the gains are 10.64%, 25.98%, and 32.09% for r equals to 3%, 5% and 7% respectively.

Figure 6 shows the early exercise frontier with different project value volatilities, σ_v. The higher the project value volatility, the higher the investment threshold value unless

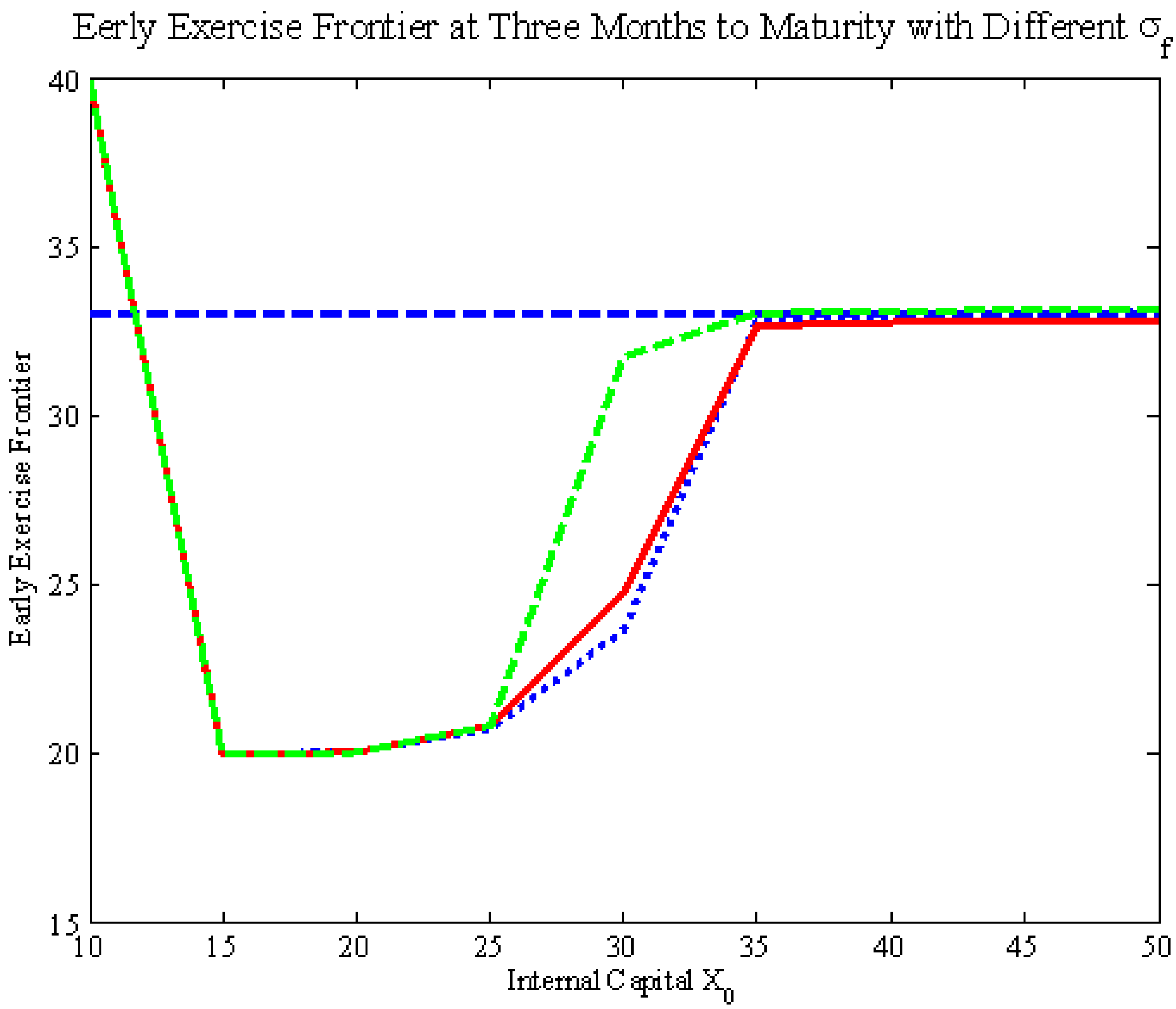

Figure 4: This figure shows the early exercise frontier at three months to maturity with different pension funding ratio volatilities. The pension funding ratio is 0.6. The horizontal dashed line is the frontier for unconstrained firm. The dashed curve is the frontier for the constrained firm with the funding ratio volatility, σ_f, equals to 0.1. The red solid curve is for $\sigma_f = 0.2$. The blue dotted curve is for $\sigma_f = 0.3$.

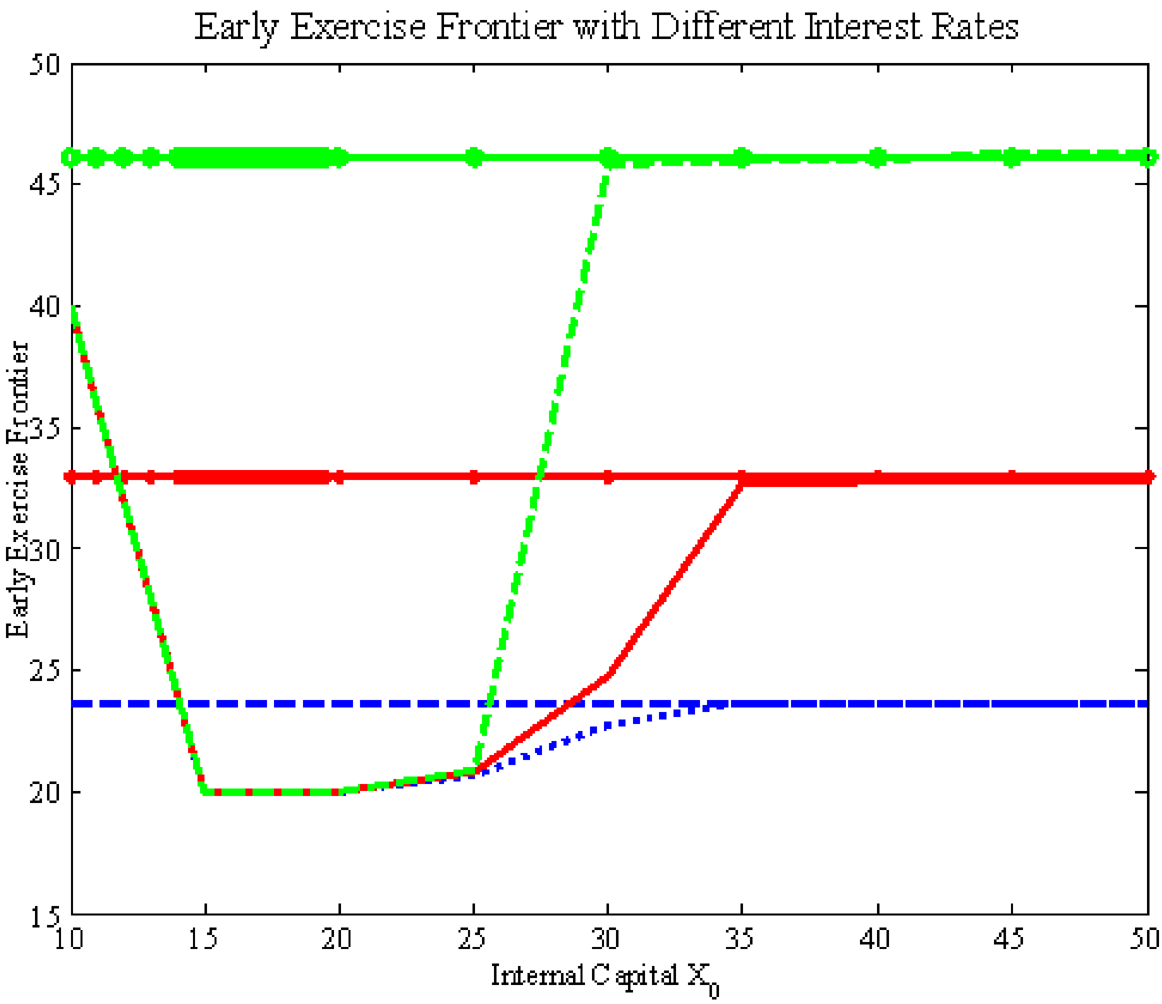

Figure 5: This figure shows the early exercise frontier at three months to maturity with different interest rates. The pension funding ratio is 0.6. The horizontal dashed line with circle is the frontier for unconstrained firm when the interest rate is 7%. The dashed curve is the frontier for the constrained firm when the interest rate equals to 7%. The red solid curve with star is for the unconstrained firm when the interest rate is 5%. The red solid line is the constrained firm when the interest rate is 5%. The blue dashed line is for the unconstrained firm when the interest rate is 3%. The blue dotted curve is for the constrained firm when the interest rate is 3%.

Table 9: This table shows gains of following a strategy taking into account of the penison plan underfunding risk with different interest rates, r. The gain is defined as the percentage difference between of two options values: one follows the investment strategy taking into account the pension plan underfunding risk and the other one following the strategy ignoring the pension plan underfunding risk. Other paramter values are presented in table 1.

	$r = 0.03$	$r = 0.05$	$r = 0.07$
10	-1.31%	2.58%	-3.57%
15	10.64%	25.98%	32.09%
16	14.50%	49.65%	45.28%
17	18.69%	57.76%	57.81%
18	20.45%	77.89%	67.36%
19	21.50%	71.74%	57.72%
20	27.44%	74.32%	54.83%
40	0.50%	0.53%	-1.07%

Table 10: This table shows gains of following a strategy taking into account of the penison plan underfunding risk with different project value volatilities, σ_v. The gain is defined as the percentage difference between of two options values: one follows the investment strategy taking into account the pension plan underfunding risk and the other one following the strategy ignoring the pension plan underfunding risk. Other parameter values are presented in table 1.

	$\sigma_v = 0.3$	$\sigma_v = 0.2$	$\sigma_v = 0.1$
10	1.99%	2.58%	−17.68%
15	26.31%	25.98%	25.35%
16	38.05%	49.65%	30.70%
17	46.83%	57.76%	43.48%
18	57.48%	77.89%	60.15%
19	60.06%	71.74%	75.46%
20	59.01%	74.32%	77.09%
40	0.94%	0.53%	−0.09%

the firm's exercise frontier is determined by the borrowing constraint: higher project value volatility means higher chances to have high project values in the future and, therefore, leads to higher value of waiting. Table 10 shows the gains of adopting the investment strategy proposed by this chapter. No matter what the project value volatilities are, the gains are large. For example, when the amount of cash the firm has is 18, the gains are 57%, 78% and 60% for $\sigma_v = 0.3$, 0.2 and 0.1 respectively.

Table 11 shows the option values with underfunded pension plans when the diffusion processes of cash flow and the investment project value, ρ_{XV}, are either positively (70%) or negatively (-70%) correlated and the correlation between cash flow and pension funding ratio keeps at 0. The investment option values are increasing with ρ_{XV}. For example, when the initial cash flow is 15, the option value for a firm where ρ_{XV} is -70% is 0.4474. The option value increases to 1.4353 when the correlation increases to 70%.

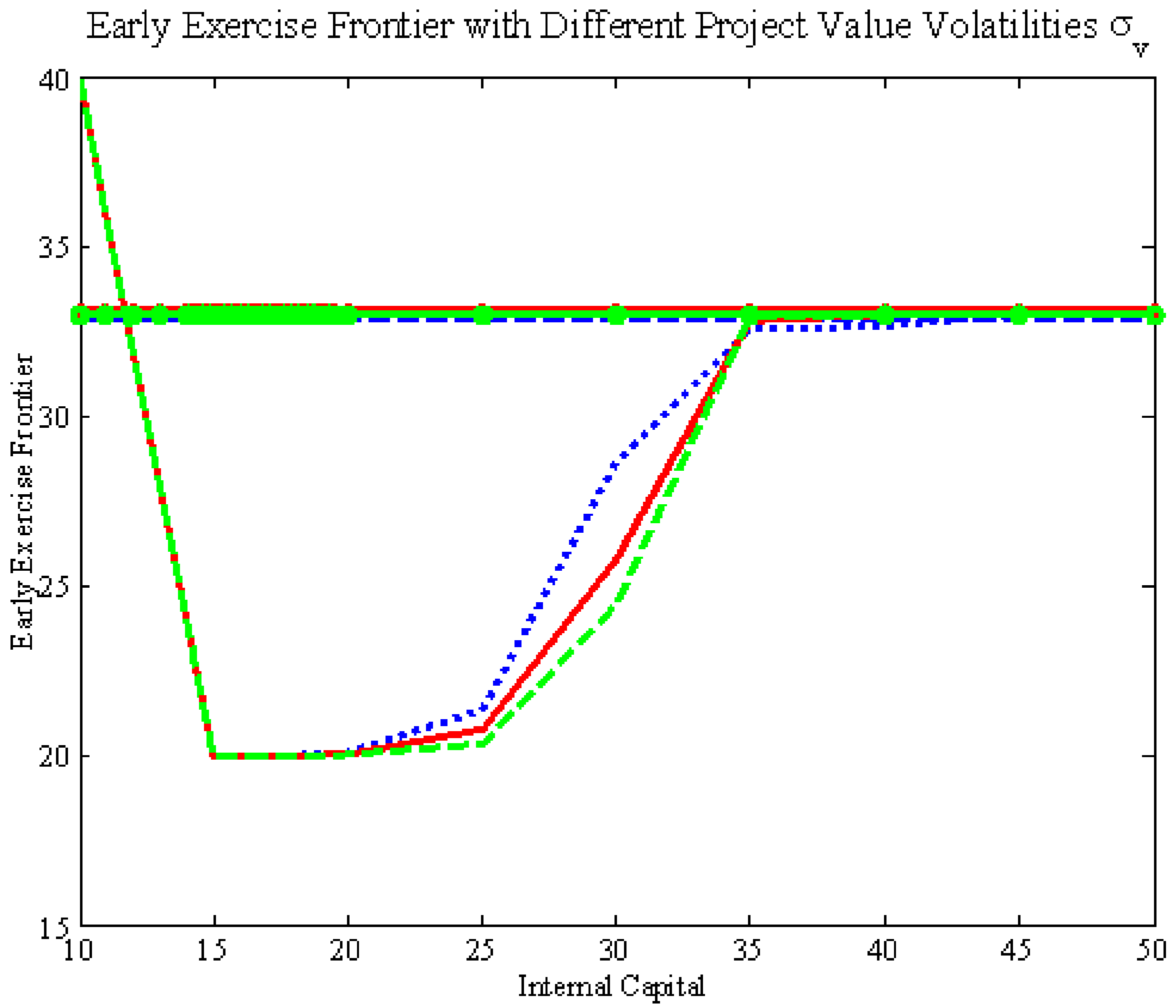

Figure 6: This figure shows the early exercise frontier at three months to maturity with different interest rates. The pension funding ratio is 0.6. The horizontal dashed line with circle is the frontier for unconstrained firm when $\sigma_v = 10\%$. The dashed curve is the frontier for the constrained firm when $\sigma_v = 10\%$. The red solid curve with star is for the unconstrained firm when $\sigma_v = 20\%$. The red solid line is the constrained firm when $\sigma_v = 20\%$. The blue dashed line is for the unconstrained firm when $\sigma_v = 30\%$. The blue dotted curve is for the constrained firm when $\sigma_v = 30\%$.

Table 11: This table shows the option value with different ρ_{XV}. Firm 2 follows the optimal investment strategy taking into account the pension underfunding risk and firm 1 follows the strategy which is optimal for a liquidity-constrained firm without a pension plan. The parameter values are presented in table 1. X_0 stands for the amount of internal capital.

Cash	**Option Value** ($\rho_{XV} = -0.7$)			**Option Value** ($\rho_{XV} = 0.7$)		
X_0	Firm 1	Firm 2	% Increase	Firm 1	Firm 2	% Increase
10	0	0	-	0.4088	0.4504	10.18%
15	0.2309	0.4474	93.71%	0.6316	1.4353	127.22%
17	0.3544	0.9954	180.85%	0.6539	1.4182	116.88%
20	0.3010	1.3189	338.17%	1.0747	1.3635	26.87%

The positive correlation means that the investment project value and the cash flow will improve together and thus, more chances that the option will be exercised in the future. Table 12 shows the option value with underfunded pension plans where the correlations between cash flow and pension plan funding ratios, ρ_{XF}, are either 70% or -70% and ρ_{XV} keeps at 0. The investment option values are increasing with ρ_{XF}. When ρ_{XF} and X_0 equal to -70% and 15 respectively, the investment option value (firm 2) is 1.0383. The investment option value (firm 2) increases to 1.2122 when ρ_{XF} equals to 70%. When the correlation between cash flow and pension funding ratio is positive, cash inflow and pension funding ratio improve together which decreases the possibilities that the firm will be liquidity constrained in the future. Following the strategy taking into account the correlations lead to large and positive gains for any correlations.

We derived an optimal investment strategy for a liquidity-constrained firm sponsoring a underfunded DB pension plan. We show that the investment strategy should depends on the pension plan's funding status. We evaluated the gain of adopting such an investment strategy by comparing the investment option value of two otherwise identical firms one follows the strategy proposed by this chapter and one follows the investment strategy ignoring the pension plan underfunding risk. We find that adopting the strategy proposed by this chapter can lead to significant increases in the investment option value. The result is robust with respect to changes in parameter values.

Table 12: This table shows the option value with different ρ_{XF}. Firm 2 follows the optimal investment strategy taking into account the pension underfunding risk and firm 1 follows the strategy which is optimal for a liquidity-constrained firm without a pension plan. The parameter values are presented in table 1. X_0 stands for the amount of internal capital.

Cash	**Option Value** ($\rho_{XF} = -0.7$)			**Option Value** ($\rho_{XF} = 0.7$)		
X_0	Firm 1	Firm 2	% Increase	Firm 1	Firm 2	% Increase
10	0.1010	0.1083	7.25%	0.1088	0.1171	7.60%
15	0.7103	1.0383	46.17%	0.8591	1.2122	41.09%
17	0.5215	1.2383	137.43%	0.7881	1.3921	76.63%
20	0.5651	1.3588	140.42%	0.7585	1.3882	83.01%

IV Conclusions

The loss a underfunded pension plan brings to the sponsoring firm is twofold. First, by law the sponsoring company has to make additional contributions to the pension plan to close the funding gap. Second, the mandatory contribution depletes the firm's capital and therefore limits its ability to conduct core business investment. We argue that a sponsoring firm should adopt an optimal investment strategy which takes into account the pension underfunding risk.

We show that in a market where borrowing is limited, the firm's optimal investment strategy depends on the amount of capital available for investment and the initial pension funding ratio. When the amount of capital for investment is very small, firms with different pension funding ratios have the same investment strategy. At low levels of capital, the investment strategy is determined by the borrowing constraint. When the amount of capital for investment is very large, firms do not have liquidity constraint and therefore follow the same strategy. When a firm is neither extremely constrained nor unconstrained, the sponsoring firm's investment threshold value for an investment project decreases when the pension plan funding ratio worsens. This is because firms with lower pension plan funding ratios have larger expected pension contributions and therefore lower values of waiting. The result indicates that firms with lower pension funding ratio should invest earlier than otherwise identical firms with higher pension funding ratio. We show that the investment option value increases significantly in response to adopt the strategy proposed by this chapter.

Appendix The Early Exercise Frontier

This appendix shows the details of the steps to obtain early exercise frontiers. Let $X_{T-\Delta t} = \overline{X}_{T-\Delta t}$, $F_{T-\Delta t} = \overline{F}_{T-\Delta t}$, and $\hat{V}^1_{T-\Delta t}$ be the initial project value, where $\overline{X}_{T-\Delta t}$ and $\overline{F}_{T-\Delta t}$ can be any elements in the vectors $\mathbf{X}_{T-\Delta t}$ and $\mathbf{F}_{T-\Delta t}$ and the initial project value $\hat{V}^1_{T-\Delta t}$ is chosen to be equal to the investment cost, I. We first calculate the value of waiting, $\eta_{T-\Delta t}\left(\overline{X}_{T-\Delta t},\ \overline{F}_{T-\Delta t}, \hat{V}^1_{T-\Delta t}\right)$. From the project value, $\hat{V}^1_{T-\Delta t}$, we can find a new project value, $\hat{V}^2_{T-\Delta t}$, which fulfills the following two conditions

$$\hat{V}^2_{T-\Delta t}\left(X_{T-\Delta t}, F_{T-\Delta t}\right) - I + X_{T-\Delta t} + E_{T-\Delta t}A_T + G_{T-\Delta t} = \eta_{T-\Delta t}\left(\overline{X}_{T-\Delta t},\ \overline{F}_{T-\Delta t}, \hat{V}^1_{T-\Delta t}\right), \tag{10}$$

$$\hat{V}^2_{T-\Delta t}\left(X_{T-\Delta t}, F_{T-\Delta t}\right) \geq \frac{I - X_{T-\Delta t}}{c}, \tag{11}$$

where the left side of (10) stands for the value of immediate exercise when the project value is $\hat{V}^2_{T-\Delta t}\left(X_{T-\Delta t}, F_{T-\Delta t}\right)$, and the right side of (10) is the option value of waiting when the project value is $\hat{V}^1_{T-\Delta t}$, while (11) requires that the liquidity constraint is fulfilled. By using the same steps, starting from project value $\hat{V}^2_{T-\Delta t}$ we can find a new project value, $\hat{V}^3_{T-\Delta t}$. We will repeat the same procedure many times until the project value converges, that is, the difference between the outcomes of two simulations is very small, $\left|\hat{V}^{n+1}_{T-\Delta t} - \hat{V}^n_{T-\Delta t}\right| \leq \epsilon$, $\epsilon = 0.0001$. $\hat{V}^n_{T-\Delta t}$ is the investment threshold at time $T - \Delta t$ when the value of liquid assets is $\overline{X}_{T-\Delta t}$ and the funding ratio is $\overline{F}_{T-\Delta t}$. We calculate the corresponding investment threshold values for all combinations of elements in the vectors $\mathbf{X}_{T-\Delta t}$ and $\mathbf{F}_{T-\Delta t}$. From the grid $\left\{\left(X_{T-\Delta t,1}, F_{T-\Delta t,1}, \hat{V}_{T-\Delta t,1}\right), \left(X_{T-\Delta t,1}, F_{T-\Delta t,2}, \hat{V}_{T-\Delta t,2}\right),\ \ldots\left(X_{T-\Delta t,m},\right.\right.$ $\left.\left.F_{T-\Delta t,m},\quad \hat{V}_{T-\Delta t,m},\right)\right\}$ we then estimate a function, $F_{T-\Delta t}\left(\mathbf{X}_{T-\Delta t},\ \mathbf{F}_{T-\Delta t}\right)$, which represents the early exercise frontier at time $T - \Delta t$. $F_{T-\Delta t}\left(\mathbf{X}_{T-\Delta t},\ \mathbf{F}_{T-\Delta t}\right)$ is estimated by regressing $\hat{\mathbf{V}}_{T-\Delta t}$ on polynomials in $X'_{T-\Delta t}s$ and $F'_{T-\Delta t}s$. After obtaining the parameterized exercise frontier at time $T - \Delta t$, we proceed backwards using the same procedure.

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